Chapter 1

Moving coordinate system

1.1 Introduction

Moving coordinate systems are important because, no material body is at absolute rest. As we know, even galaxies are not stationary. Therefore, a coordinate frame at absolute rest is hypothetical, hypothesized by Newton, where his laws of motion hold. In reality, we have the moving frames, prime example being Earth itself. We therefore need to know how the Newton's laws operate in a moving frame like a rotating frame (e.g. Earth).

1.2 Rectilinear moving frame

Let us consider a simplest case of a rectilinear motion. Let the frame S' move relative to S. For simplicity, let us assume that, the axes of S and S' are parallel. The instantaneous position of O' relative to the origin O is depicted by $\vec{R}(t)$ in figure 1.1. Consider a rectilinear motion of S' along the direction of $\vec{R}(t)$. Then vector $\vec{r}$ in S will be $\vec{r}\,'$ in S' such that,

$$\vec{r}\,' = \vec{r} - \vec{R}(t) \tag{1.1}$$

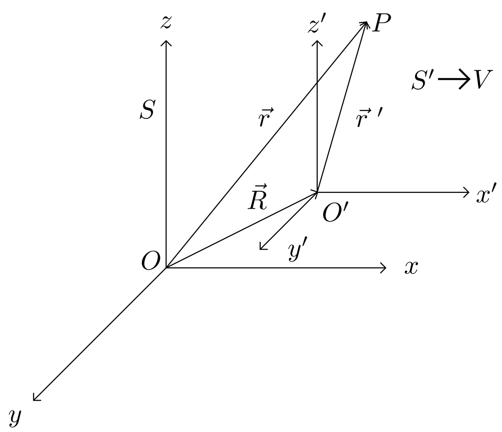

Figure 1.1: Frame S', moving relative to S along vector $R(t)$.

The time derivative of the equation is

$$\dot{\vec{r}}\,' = \dot{\vec{r}} - \dot{\vec{R}}(t) \tag{1.2}$$

$\vec{v}\,'$ is the velocity in S' and is related with velocity $\vec{v}$ in S as,

$$\vec{v}\,' = \vec{v} - \vec{V} \tag{1.3}$$

where $V = \dot{\vec{R}}(t)$ is velocity of S' relative to S. Let us assume that, the Newton's second law is valid in S. Then, the external force F_{ext} is

$$\begin{aligned} F_{ext} &= m\ddot{\vec{r}} = m\frac{d\vec{v}}{dt} \\ &= m\left(\frac{d\vec{v}\,'}{dt} + \frac{d\vec{V}}{dt}\right) = m\left(\ddot{\vec{r}}\,' + \frac{d\vec{V}}{dt}\right) \end{aligned} \tag{1.4}$$

If S' moves relative to S with constant velocity, i.e., $dV/dt = 0$, then S and S' are indistinguishable.

$$F_{ext} = m\ddot{\vec{r}} = m\ddot{\vec{r}}\,' \tag{1.5}$$

Thus, S and S' are inertial. We call the inertial force measured in the fixed frame S as $F = m\ddot{\vec{r}}$. Then, the inertial force measured in S' is $F' = m\ddot{\vec{r}}\,'$. The F_{ext} is independent of motion of S and S'. If $\dot{\vec{V}} \neq 0$,

$$F' = m\ddot{\vec{r}} - m\ddot{\vec{R}} = F - m\ddot{\vec{R}} \tag{1.6}$$

It must be noted that if S' is moving with respect to S with constant velocity, the law of force has a same form. (In technical jargon, it is covariant). However, if S' is an accelerated frame, then it can be distinguished from S by virtue of the term $m\ddot{\vec{R}}$. Some times, the term $m\ddot{\vec{R}}$ of S' is called fictitious force. This is a misnomer because, we get a jerk when a train starts.

1.3 Rotating frame of reference

Consider two frame of reference S and S' with unit vectors $n = (i, j, k)$ and $n' = (i', j', k')$ and with a common origin. S' rotates with some axis with angular velocity $\vec{\omega}$. Let us consider a position vector $\vec{r}$. The components of $\vec{r}$ in S are (x, y, z) and in S' are (x', y', z'). Then,

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \tag{1.7}$$
$$\vec{r} = \hat{i}'x' + \hat{j}'y' + \hat{k}'z' \tag{1.8}$$

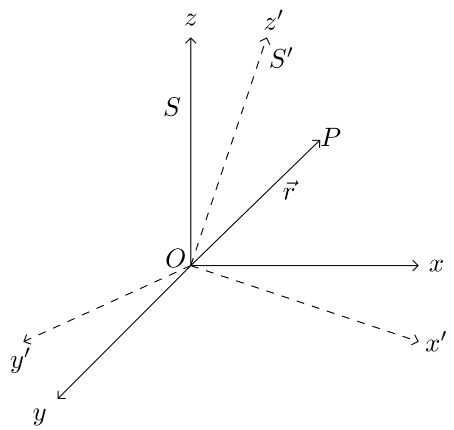

Figure 1.2: Vector $\bar{r}$ in S and S'.

As S' rotates, unit vectors $n' = (i', j', k')$ are functions of time. Let us write d/dt as a time derivative operator in S and d'/dt as a time derivative operator in S'. We do not write d'/dt' because we assume $t = t'$, a non relativistic case. Since S is a fixed frame, (i, j, k) do not change with time. Clearly,

$$v = \frac{d\vec{r}}{dt} = \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z} \tag{1.9}$$

The time derivative of the equation 1.8 is

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \hat{i}'\frac{dx'}{dt} + \frac{d\hat{i}'}{dt}x' + \hat{j}'\frac{dy'}{dt} + \frac{d\hat{j}'}{dt}y' + \hat{k}'\frac{dz'}{dt} + \frac{d\hat{k}'}{dt}z' \\
\frac{d\vec{r}}{dt} &= \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt} + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt} \\
v &= v' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt} \qquad (1.10)
\end{aligned}$$

where

$$v' = \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt} = \frac{d}{dt}(\hat{i}'x' + \hat{j}'y' + \hat{k}'z')$$

is the velocity in the frame S'. We need to find the value of dn'/dt in a fixed frame in terms of the known quantities like $\vec{\omega}$ and $\vec{r}$. We must prove a following important lemma.

Lemma: Let there be a general direction OM around which a vector A of constant magnitude rotates with a constant angular velocity $\vec{\omega}$ in a fixed frame. Then

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

Proof: Let the tail of the vector A be at O and let it rotate about OM with angular velocity $\vec{\omega}$ as shown in

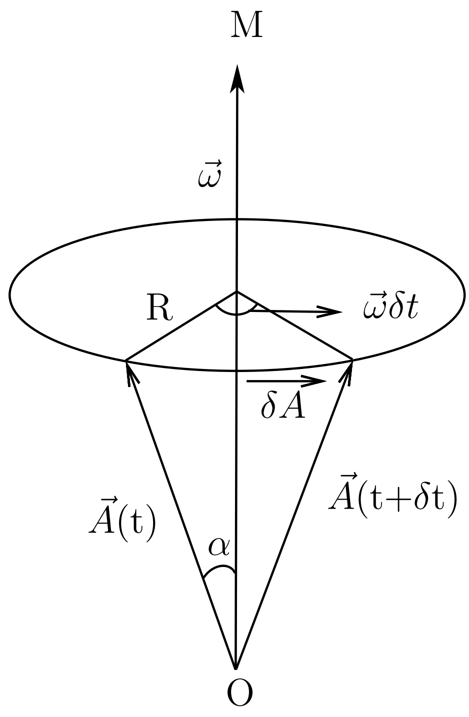

Figure 1.3: Vector $\vec{A}$ is rotating around a fixed direction OM.

the figure 1.3. The position of the vector A at time t and $t + \delta t$ is shown in the figure when it rotates in time δt. As the direction of $\vec{A}$ changes, $\vec{A(t)}$ changes to $\vec{A}(t + \delta t)$. The magnitude of $\vec{A}(t)$ and $\vec{A}(t + \delta t)$ is same but,

$$\vec{A}(t + \delta t) = \vec{A}(t) + \delta\vec{A} \qquad (1.11)$$

From the figure 1.3,

$$\mid \delta A \mid = R\,\omega\,\delta t \qquad (1.12)$$

Again form the figure 1.3,

$$R = A\; sin\,\alpha \qquad (1.13)$$

$$\begin{aligned}
\delta\vec{A} &= \omega A \, sin\,\alpha \, \delta t \\
\delta\vec{A} &= (\vec{\omega}\times\vec{A})\,\delta t \\
\frac{\delta\vec{A}}{\delta t} &= \vec{\omega}\times\vec{A} \qquad (1.14)
\end{aligned}$$

When $\delta t \to 0$, the equation 1.14 gives

$$\frac{d\vec{A}}{dt} = \vec{\omega}\times\vec{A} \qquad (1.15)$$

This derivative is taken in the frame S. In the frame where A is fixed (i.e., S' frame), $d'A/dt = 0$. And, in S', OM rotates with angular velocity $-\vec{\omega}$ i.e., in a counter clockwise direction. Equation (1.15) can be written in an operator form as,

$$\frac{d(\ )}{dt} = \vec{\omega}\times(\ ) \qquad (1.16)$$

where, ( ) contain any vector operator. Thus,

$$\begin{aligned}
\frac{d\hat{i}'}{dt} &= \vec{\omega}\times\hat{i}' \qquad (1.17)\\
\frac{d\hat{j}'}{dt} &= \vec{\omega}\times\hat{j}' \qquad (1.18)\\
\frac{d\hat{k}'}{dt} &= \vec{\omega}\times\hat{k}' \qquad (1.19)
\end{aligned}$$

Using equations 1.17, 1.18 and 1.19 in equation 1.10,

$$\begin{aligned}
v &= v' + x'(\vec{\omega}\times\hat{i}') + y'(\vec{\omega}\times\hat{j}') + z'(\vec{\omega}\times\hat{k}') \\
v &= v' + \vec{\omega}\times(\hat{i}'x' + \hat{j}'y' + \hat{k}'z') \\
v &= v' + \vec{\omega}\times\vec{r} \qquad (1.20)\\
\frac{d\vec{r}}{dt} &= \frac{d'\vec{r}}{dt} + \vec{\omega}\times\vec{r} \qquad (1.21)
\end{aligned}$$

From equation 1.21, the general operator equation can be written as,

$$\frac{d(\ )}{dt} = \frac{d'(\ )}{dt} + \vec{\omega}\times(\ ) \qquad (1.22)$$

We use equation 1.22 to get acceleration in a rotating frame in terms of its value in the fixed frame. We denote a and a' as accelerations in S and S'. Thus by operating $\vec{v}$ to the equation 1.22,

$$\frac{d\vec{v}}{dt} = \frac{d'\vec{v}}{dt} + \vec{\omega}\times\vec{v} \qquad (1.23)$$

$$\begin{aligned}
\frac{dv}{dt} &= \frac{d'}{dt}(\vec{v}\,' + \vec{\omega}\times\vec{r}) + \vec{\omega}\times(\vec{v}\,' + \vec{\omega}\times\vec{r}) \\
a &= \frac{d'\vec{v}\,'}{dt} + \vec{\omega}\times\frac{d'\vec{r}}{dt} + \frac{d'\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\vec{v}\,' + \vec{\omega}\times\vec{\omega}\times\vec{r} \\
a &= a' + 2(\vec{\omega}\times v') + \vec{\omega}\times\vec{\omega}\times\vec{r} + \frac{d'\vec{\omega}}{dt}\times\vec{r} \qquad (1.24)
\end{aligned}$$

It must be noted that, the time derivative of a vector parallel to ω is same in both S and the S' frames. Thus,

$$\frac{d'\vec{\omega}}{dt} = \frac{d\vec{\omega}}{dt}$$

If F_{ext} is the external force, according to Newton's second law of motion, valid in the fixed frame as,

$$F_{ext} = ma \qquad (1.25)$$

Using equation 1.24,

$$F_{ext} = ma = ma' + 2m(\vec{\omega}\times v') + m(\vec{\omega}\times\vec{\omega}\times\vec{r}) + m(\dot{\vec{\omega}}\times\vec{r}) \qquad (1.26)$$

As everyone is on the surface of Earth, everyone is in a rotating frame, he measures $\vec{v}\,'$ and $\vec{a}\,'$ and not v and a. Equation 1.26 for an observer in rotating frame as,

$$ma' = F_{ext} - 2m(\vec{\omega}\times v') - m(\vec{\omega}\times\vec{\omega}\times\vec{r}) - m(\dot{\vec{\omega}}\times\vec{r}) \qquad (1.27)$$

Equation 1.27 gives a motion of a particle in the rotating frames. If ω is constant, and it is the case in many situations, the last term $-m(\dot{\vec{\omega}} \times \vec{r} = 0)$, then,

$$ma' = F_{ext} - 2m(\vec{\omega} \times v') - m(\vec{\omega} \times \vec{\omega} \times \vec{r}) \quad (1.28)$$
$$F' = F_{ext} + F_c + F_r = F_{ext} + F_o \quad (1.29)$$

where $F_c = 2m(\vec{\omega} \times v')$ is called Coriolis force and $F_r = -m(\vec{\omega} \times \vec{\omega} \times \vec{r})$ is called centrifugal force. In a fixed frame, the term $m(\vec{\omega} \times \vec{\omega} \times \vec{r})$ is called a centripetal force, directed towards the center and its magnitude is $m\omega^2 r = mv^2/r$. In a rotating frame, we have a minus sign in front of the centripetal force, directed outwards and is called centrifugal force.
$F_o = F_c + F_r$ is not real force. It is called fictitious force. It is not present in a fixed coordinate system. We can treat a rotating coordinate system as if it were fixed by adding a centrifugal force and a Coriolis force. If we fix a coordinate system with the rotating particle, then it is at rest in this frame. The centripetal force is balanced by centrifugal force in this frame.
Coriolis force depends on the velocity of the particle and it acts in a direction perpendicular to $\vec{v}$. Therefore, Coriolis force does no work but only changes its direction of motion.
Exercise: Estimate the magnitude of

1. Angular velocity of earth
2. Centrifugal acceleration at equator
3. Coriolis acceleration at latitude of 45^o with velocity $10^3 ms^{-1}$

1.4 Applications

1.4.1 Effect of centrifugal force on acceleration due to gravity

The acceleration due to gravity varies with latitude ϕ, being about 0.5% smaller at equator than its value at the poles. This is the reason why our Earth has oblate shape i.e., Earth's sphere of flattened at the poles.
Consider a point P with latitude ϕ as shown in figure 1.4. If Earth was not rotating, g at P would act along PO. Centrifugal acceleration is to be superposed on g to get g_{eff}. Thus,

$$g_{eff} = g - \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (1.30)$$

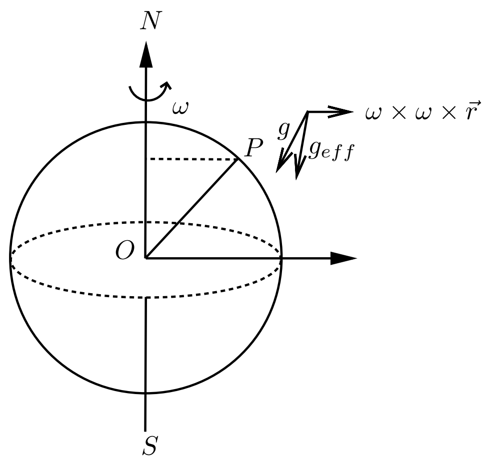

Figure 1.4: Direction of effective acceleration due to gravity

Direction of g_{eff} does not pass through O, the centre of Earth. The direction of plumb line is along g_{eff} as shown in figure 1.4. Earth's surface is flattened to such an extent that is always perpendicular to g_{eff}

1.4.2 Effect of Coriolis force on atmospheric air flow

It is observed that, there is not much of an air flow in vertical direction as compared with a horizontal direction. This is so because, in a vertical direction, pressure decrease as one moves upwards and this force is balanced by weight of the air parcel. This leads to a persistent long range motion of air in the form of winds.

In the absence of Coriolis force, the air flow takes place from higher pressure to lower pressure. Let ω be perpendicular to the plane. The Coriolis acceleration $(-2\omega \times v)$ deflects air current as shown in figure 1.5. The wind current then circulates around a low pressure zone in clockwise direction.

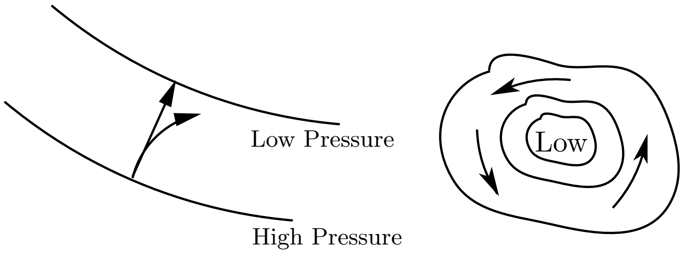

Figure 1.5: Cyclone formation due to Coriolis force

1.4.3 Foucault pendulum

French physicist Foucault realized that, the Coriolis force would rotate the plane of oscillation of pendulum. Foucault pendulum is an ordinary pendulum with a large length as much as more than 10 meter.
Let us assume that a pendulum is suspended from the origin O' of a frame S' at a place of latitude ϕ, as shown in figure 1.6. The $Z-axis$ of S' is vertically upward, the $X-axis$ due south and the $Y-axis$ due east, but with the origin at a height above the surface of the earth at P. The position vector of the bob B is $\vec{r} \equiv (x, y, z)$ and $O'B = l = \vec{r}$ is the length of the pendulum. The tension in the string is the applied force F_a and since it is along BO', we may write,

$$\begin{aligned} F_a &= k_1^2 m\vec{r} \\ \ddot{\vec{r}} &= k_1^2 \vec{r} \end{aligned} \tag{1.31}$$

where $k_1^2 = g/l$ is a constant of proportionality. Therefore, the equation of motion of the pendulum is,

$$\ddot{\vec{r}} - k_1^2 \vec{r} = 0 \tag{1.32}$$

The equation 1.31 can be satisfied only when there is no other force acting on the pendulum except gravitational force. If the rotation of Earth is taken into account, $g > g_{eff}$ and in addition a Coriolis force act on the pendulum. Thus, the equation of pendulum becomes,

$$\ddot{\vec{r}} - k^2 \vec{r} = -2\vec{\omega} \times \vec{v} \tag{1.33}$$

where $k^2 = g_{eff}/l$. Since x-axis due south and the Y-axis due to east, the vector $\vec{\omega}$ is in the ZX-plane, so that

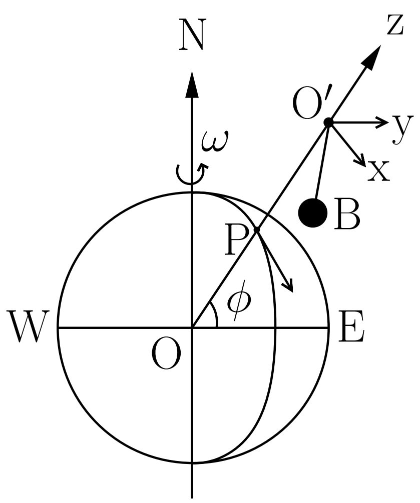

Figure 1.6: Position of the Foucault's pendulum on the surface of the Earth.

$\omega_y = 0$ and we have

$$\omega_x = -\omega cos\phi, \qquad \omega_y = 0 \qquad \omega_z = \omega sin\phi, \tag{1.34}$$

Since motion of the pendulum is in the XY-plane, so that $v_z = 0$. Then clearly

$$\vec{\omega} = (-\omega cos\phi, 0, \omega\, sin\phi) \tag{1.35}$$
$$\ddot{\vec{r}} = (\ddot{x}, \ddot{y}, 0) \tag{1.36}$$

Resolving the equation of motion in components,

$$\ddot{x} - k^2 x = -2(\vec{\omega} \times \vec{v})_x = 2\omega\dot{y}\, sin\phi \tag{1.37}$$
$$\ddot{y} - k^2 y = -2(\vec{\omega} \times \vec{v})_y = -2\omega(\dot{x}\, sin\phi + \dot{z}\, cos\phi) \tag{1.38}$$
$$\ddot{z} - k^2 z + g = -2(\vec{\omega} \times \vec{v})_z = 2\omega\dot{y}\, cos\phi \tag{1.39}$$

For small oscillations, $z = -l$ and therefore $\dot{z} = \ddot{z} = 0$. Again, since $g \approx 1000\,\text{cms}^{-2}$ and $\omega = 7.3 \times 10^{-5} rad/s$, the term containing ω in equation 1.39 may be neglected in comparison with g and therefore we obtain $k^2 = -g/l$. Then equations 1.37 and 1.38 may be written as

$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\, sin\phi \tag{1.40}$$
$$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x}\, sin\phi \tag{1.41}$$

Introducing the complex variable $z = x + iy$ and writing $u = \omega sin\phi$, equations 1.40 and 1.41 may be written as a single equation

$$\ddot{z} + 2iu\dot{z} + \frac{g}{l}z = 0 \tag{1.42}$$

[1]

Solution of the equation 1.42 is in the form $z = e^{i\alpha t}$, where α is a constant and on substitution to the equation 1.42,

$$\alpha^2 + 2u\alpha - \frac{g}{l} = 0 \tag{1.43}$$

The roots of the equation 1.43 are,

$$\alpha_1 = -u + \sqrt{u^2 + \frac{g}{l}} \qquad \alpha_2 = -u - \sqrt{u^2 + \frac{g}{l}} \tag{1.44}$$

Thus general solution of the equation 1.42 is

$$z = Ae^{i\alpha_1 t} + Be^{i\alpha_2 t} \tag{1.45}$$
$$\dot{z} = i\alpha_1 Ae^{i\alpha_1 t} + i\alpha_2 Be^{i\alpha_2 t} \tag{1.46}$$

where A and B are constants to be determined by the initial conditions. If the pendulum is drawn due south (along the X-axis) at distance a and let go with zero initial velocity in the ZX-plane, we have, as initial conditions, $z = (x + iy) = (a + 0) = a, \dot{z} = 0$ at $t = 0$. Equation 1.45 and 1.46 yeilds,

$$a = A + B \tag{1.47}$$
$$0 = \alpha_1 A + \alpha_2 B, \qquad -\alpha_1 A = \alpha_2 B \tag{1.48}$$

Then from equation 1.45,

$$\begin{aligned} \dot{z} &= iA\alpha_1(e^{\alpha_1 t} - e^{i\alpha_2 t}) \\ &= iA\alpha_1 \left[e^{i(-u+\sqrt{u^2+\frac{g}{l}})t} - e^{i(-u-\sqrt{u^2+\frac{g}{l}})t}\right] \\ &= iA\alpha_1 e^{-iut}\left[e^{i(\sqrt{u^2+\frac{g}{l}})t} - e^{-i(\sqrt{u^2+\frac{g}{l}})t}\right] \end{aligned}$$
$$\dot{z} = -2A\alpha_1 e^{-iut} sin\left(\sqrt{u^2 + \frac{g}{l}}t\right) \tag{1.49}$$

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[1]

$$z = x + iy, \qquad \dot{z} = \dot{x} + i\dot{y}, \qquad \ddot{z} = \ddot{x} + i\ddot{y} = -\frac{g}{l}x + 2\omega\dot{y}\, sin\phi - i\frac{g}{l}y - i2\omega\dot{x}\, sin\phi$$
$$\ddot{z} = -\frac{g}{l}(x + iy) - 2i\omega\, sin\phi(\dot{x} + i\dot{y}) = -\frac{g}{l}z - 2i\omega\, sin\phi\, \dot{z}$$

As we know, the velocity of the bob becomes zero every time the pendulum reverses its direction and at these values of t, we have $\dot{z} = 0$, and this happens when the sine function is zero and therefore for values of t for which

$$\sqrt{u^2 + \frac{g}{l}}t = n\pi; \qquad (n = 0, 1, 2, 3, ...)$$

For one oscillation $t = T$ and $n = 2$,

$$\sqrt{u^2 + \frac{g}{l}}T = 2\pi; \qquad T = \frac{2\pi}{\sqrt{u^2 + \frac{g}{l}}} \tag{1.50}$$

where $u = \omega\, sin\phi$. If ω is negligible, $u \approx 0$ and we obtain $T = \dfrac{2\pi}{g/l}$. If $u \neq 0$,

$$\sqrt{u^2 + \frac{g}{l}} = \frac{2\pi}{T} \tag{1.51}$$

Using equation 1.51 in 1.44,

$$\alpha_1 = -u + \frac{2\pi}{T} \qquad \alpha_2 = -u - \frac{2\pi}{T} \tag{1.52}$$

Rewriting the equation 1.45 using 1.52,

$$z(t) = Ae^{i\left(-u+\frac{2\pi}{T}\right)t} + Be^{i\left(-u-\frac{2\pi}{T}\right)t}$$
$$z(t) = e^{-iut}\left(Ae^{\frac{i2\pi t}{T}} + Be^{\frac{-i2\pi t}{T}}\right) \tag{1.53}$$

When the pendulum is initially set in motion by drawing it southward along the X-axis to the position P_1 with a distance a, we have $t = 0$ and therefore, we see from equation 1.53 that

$$z(0) = A + B = a \tag{1.54}$$

After half an oscillation, $t = T/2$, the bob is at the northern end at P_2 and we have

$$z(T/2) = e^{-iuT/2}(-A - B) = -ae^{-iuT/2} = ae^{i(\pi - uT/2)} \tag{1.55}$$

Equation 1.55 shows that, the vector $Z(0)$ representing the point P_1 has been rotated into the position P_2 through an angle $(\pi - uT/2)$. This means that the trajectory of the pendulum, in the first half oscillation is the curve from P_1 to P_2 as shown in figure 1.7. Similarly, when $t = T$, we have

$$z(T) = e^{-iuT}(A + B) = ae^{-iuT} = z(0)e^{-iuT} \tag{1.56}$$

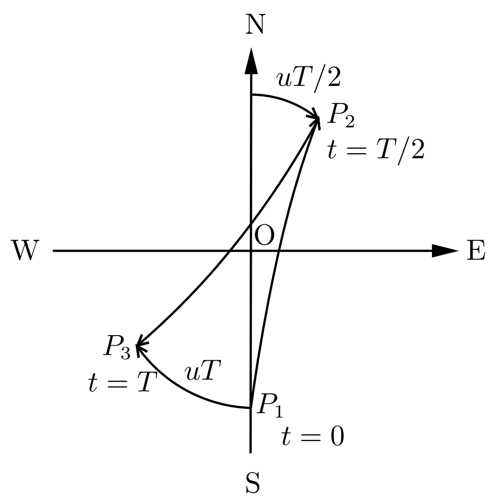

Figure 1.7: Rotation of the plane of oscillation of Foucault's pendulum.

Equation 1.56 shows that after one complete oscillation, in time T, the position of the bob of the pendulum is again at the southern end but rotated through an angle $-uT$ into position P_3. Thus the trajectory in the second half oscillation is the curved path from P_2 to P_3 as shown in the figure. We thus see that the motion of the bob is as if the plane of vibration of the pendulum rotated clockwise through an angle uT in one period. Thus the time required for one complete rotation of the plane, ie., though an angle 2π is

$$\tau = \frac{T}{uT}2\pi = \frac{2\pi}{\omega\, sin\phi} = \frac{24}{sin\phi}\text{hours} \tag{1.57}$$

since $\frac{2\pi}{\omega} = 1day = 24$ hours. Thus, at a place of latitude $\phi = 45^o$, the time τ for one complete rotation is $24\sqrt{2} \approx 34$ hours.

We note that the curved path from P_1 to P_2 of the bob of the pendulum is due to the force of Coriolis acting to the right of the velocity vector as the bob moves from south to north. For the same reason, the force of Coriolis acting to the right of the velocity as the bob moves from north to south accounts for the part of the trajectory from P_2 to P_3. We recall that the other inertial force, namely the centrifugal force has already been taken care of in defining, the weight of a material point so that the trajectory P_1 to P_2 to P_3 of the bob is solely due to the force of Coriolis. Foucault carried out his experiments in 1851 in Paris, but his experiments only confirmed the rotation of the earth qualitatively. Quantitative confirmation came only in the year 1879 by the work of Kamerlingh Onnes of low temperature fame.

Reference books:

- Classical Mechanics - P V Panat
- Classical Mechanics - K N Srinivasa Rao

Chapter 2

MECHANICS OF A SYSTEM OF PARTICLES

2.1 Conservation of linear momentum and angular momentum

Consider a system of n particles of masses $m_1, m_2, m_3, ..., m_n$ at respective positions $r_1(t)$, $r_2(t)$, $r_3(t)$, ... , $r_n(t)$ at a time t. Then, the total momentum $\vec{P}$ of the system is,

$$\vec{P} = \sum_{i=1}^{n} \vec{p_i} = \sum_{i=1}^{n} m_i \vec{v}_i = \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \vec{r}_i\right) \tag{2.1}$$

The force acting on the i^{th} particle of the system has two parts (i) external force $F_i^{(e)}$, (from our side) and (ii) internal force, F_{ji} (internal force on the i^{th} particle due to the j^{th} particle). Thus, the equation of motion (Newton's second law) for the i^{th} particle is written as

$$\frac{d\vec{p}_i}{dt} = \sum_{j=1}^{n-1} \vec{F}_{ji} + \vec{F}_i^{(e)} \tag{2.2}$$

$j \neq i$, because a particle cannot exert any force on itself. Clearly, from equation 2.2, the total force acting on the system is

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\sum_i \vec{p}_i = \sum_i \sum_j \vec{F}_{ji} + \sum_i \vec{F}_i^{(e)} \tag{2.3}$$

If the Newton's third law is valid in the system, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\sum_i \sum_j \vec{F}_{ji} = 0$, the equation 2.3 becomes,

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \frac{d}{dt}\sum_i \vec{p}_i && (2.4) \\ &= \frac{d}{dt}\sum_i m_i \vec{v}_i \\ &= \frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{(e)} = F_{ext} && (2.5) \end{aligned}$$

We define a vector $\vec{R}$ as the average of the radii vectors of the particles, weighted in proportion to their mass,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{\sum_i m_i \vec{r}_i}{M} \tag{2.6}$$

The vector $\vec{R}$ defines a point known as the *center of mass*. The equation 2.5 reduces to

$$\frac{d\vec{P}}{dt} = M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = F_{ext} \tag{2.7}$$

The equation 2.5 states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. If $F_{ext} = 0$, the total linear momentum of the system,

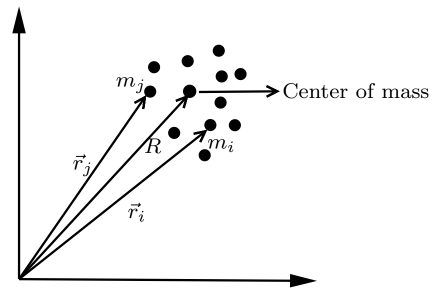

Figure 2.1: The center of mass of a system of particles.

$$\begin{aligned} \frac{d\vec{P}}{dt} &= M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = 0 \\ \vec{P} &= M\frac{d\vec{R}}{dt} = Constant \end{aligned} \tag{2.8}$$

That is total linear momentum of the system (*total mass of the system times the velocity of the center of mass*) is constant. Thus the conservation theorem for the linear Momentum of a system of particles is stated as, ***if the total external force acting on the system is zero, the total linear momentum is conserved***.
The angular momentum of the system of particles is

$$L = \sum_i \vec{r}_i \times \vec{p}_i \tag{2.9}$$

The torque acting on the system is

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) \\ &= \sum_i \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} \\ &= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_i \vec{r}_i \times \sum_j^{n-1} \vec{F}_{ij} \end{aligned} \tag{2.10}$$

If the Newton's third law is valid in the system, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\sum_i \sum_j \vec{F}_{ij} = 0$, the equation 2.10 becomes,

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} \\ \frac{d\vec{L}}{dt} &= \sum_i \vec{N}_i^{(e)} = N_{ext} \end{aligned} \tag{2.11}$$

If $N_{ext} = 0$, the total angular momentum of the system is

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \sum_i \vec{N}_i^{(e)} = 0 \\ \vec{L} &= Constant \end{aligned}$$

That is total angular momentum of the system is constant. Thus the conservation theorem for the angular momentum of a system of particles is stated as ***the angular momentum of the system is constant in time, if the applied (external) torque is zero***.

2.1.1 Theorem:

Angular momentum of a system of particles about a general origin O is equal to the angular momentum of the system concentrated at the CM plus the angular momentum of the system about its CM.
Proof: Consider a particle whose coordinate is $\vec{r}_i$ with respect to O and $\vec{r}_i\,'$ with respect to the CM. Thus

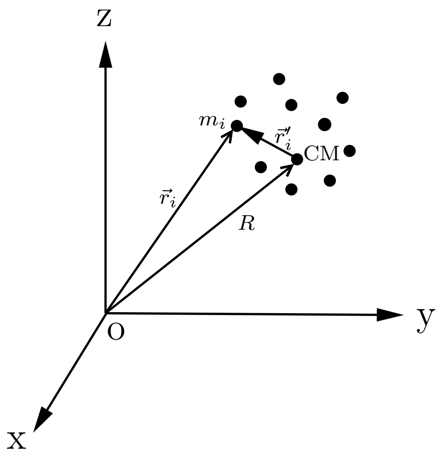

Figure 2.2: Position of i^{th} particle.

$$\begin{aligned} \vec{r}_i &= \vec{R} + \vec{r}_i\,' && (2.12)\\ \vec{v}_i &= \vec{V}_{CM} + \vec{v}_i\,' && (2.13)\\ m_i\vec{v}_i &= m_i\vec{V}_{CM} + m_i\vec{v}_i\,' \\ \vec{p}_i &= m_i\vec{V}_{CM} + \vec{p}_i\,' && (2.14)\end{aligned}$$

where, $\vec{p}_i\,' = m_i\vec{v}_i\,'$ is a velocity of i^{th} particle with reference to the *CM*. Then,

$$\begin{aligned} \vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i \\ &= \sum_i \left(\vec{R} + \vec{r}_i\,'\right) \times \left(m_i\vec{V}_{CM} + \vec{p}_i\,'\right) \\ \vec{L} &= \vec{R} \times \vec{V}_{CM}\sum_i m_i + \vec{R} \times \sum_i \vec{p}_i\,' + \sum_i (m_i\vec{r}_i\,') \times \vec{V}_{CM} + \sum_i (\vec{r}_i\,' \times \vec{p}_i\,') && (2.15)\end{aligned}$$

If CM is taken taken as the origin,

$$\sum_i (m_i\vec{r}_i\,') = 0 = \sum_i \vec{p}_i\,' \qquad (2.16)$$

The equation 2.15 reduces to,

$$\begin{aligned} \vec{L} &= \vec{R} \times \vec{V}_{CM}\sum_i m_i + \sum_i (\vec{r}_i\,' \times \vec{p}_i\,') \\ \vec{L} &= \vec{R} \times \vec{P} + \sum_i (\vec{r}_i\,' \times \vec{p}_i\,') && (2.17)\end{aligned}$$

$\vec{R} \times \vec{P}$ is an angular momentum of the centre of mass with respect to O, and $\sum_i (\vec{r}_i\,' \times \vec{p}_i\,')$ is an angular momentum of the system of particles with respect to the *CM*. From the above equation it is obvious that, $\vec{L}$ depends upon the choice of origin. If the *CM* of the system of particles is stationary, then $\vec{L}$ is independent of the choice of origin.

2.2 Energy

The work done by all forces in moving the system from an initial configuration *1*, to a final configuration *2* is

$$\begin{aligned} \int_1^2 W_{12}dW &= \sum_i \int_1^2 \vec{F}_i.d\vec{r}_i \\ \int_1^2 W_{12}dW &= \sum_i \int_1^2 \vec{F}_i^{(e)}.d\vec{r}_i + \sum_i \sum_j^{n-1} \int_1^2 \vec{F}_{ij}.d\vec{r}_{ij} && (2.18)\end{aligned}$$

If the Newton's third law is valid in the system, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\sum_i \sum_j \vec{F}_{ij} = 0$, the equation 2.18 becomes,

$$\begin{aligned}
T_2 - T_1 &= \sum_i \int_1^2 \vec{F}_i^{(e)}.d\vec{r}_i \\
&= \sum_i \int_1^2 \frac{d\vec{p}_i}{dt}.\frac{d\vec{r}_i}{dt}dt \\
&= \sum_i \frac{1}{m_i}\int_1^2 \frac{d\vec{p}_i}{dt}.\vec{p}_i dt \\
&= \sum_i \frac{1}{m_i}\int_1^2 \vec{p}_i \; d\vec{p}_i \\
T_2 - T_1 &= \left[\sum_i \frac{\vec{p}_i^{\,2}}{2m_i}\right]_2 - \left[\sum_i \frac{\vec{p}_i^{\,2}}{2m_i}\right]_1 \\
T &= \frac{1}{2}\sum_i m_i v_i^2 \qquad (2.19)
\end{aligned}$$

In center of mass coordinates, $\vec{v}_i = \vec{V}_{CM} + \vec{v}_i\,'$ and $\vec{v}_i^{\,2} = (\vec{V}_{CM} + \vec{v}_i\,').(\vec{V}_{CM} + \vec{v}_i\,')$,

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i(\vec{V}_{CM} + \vec{v}_i\,').(\vec{V}_{CM} + \vec{v}_i\,') \\
&= \frac{1}{2}\sum_i m_i\vec{V}_{CM}^2 + \vec{V}_{CM}\sum_i m_i\vec{v}_i\,' + \frac{1}{2}\sum_i m_i\left(\vec{v}_i\,'\right)^2
\end{aligned}$$

If CM is taken taken as the origin, $\sum_i \vec{p}_i\,' = \sum_i m_i\vec{v}_i\,' = 0$,

$$T = \frac{1}{2}MV_{CM}{}^2 + \frac{1}{2}\sum_i m_i\left(\vec{v}_i\,'\right)^2 \qquad (2.20)$$

Thus the kinetic energy of a system of particles is equal to the sum of the kinetic energy of the *CM* plus the kinetic energy of the system about its *CM*.
If the particle moves from initial configuration *1*, to a final configuration *2* under the action of a conservative force, then the external forces are derivable in terms of the gradient of a potential, the first term of the equation 2.18 can be written as [1]

$$\sum_i \int_1^2 \vec{F}_i^{(e)}d\vec{r}_i = -\sum_i \int_1^2 \nabla_i V_i d\vec{r}_i \qquad (2.21)$$

The internal forces are also conservative, then the mutual forces between the i^{th} and j^{th} particles, F_{ij} and F_{ji} can be obtained from a potential function V_{ij}. The seond term of the equation 2.18 can be written as

$$\sum_i\sum_j \int_1^2 \vec{F}_{ij}d\vec{r}_{ij} = -\sum_i\sum_j \int_1^2 (\nabla_i V_{ij}d\vec{r}_i + \nabla_j V_{ij}d\vec{r}_j) \qquad (2.22)$$

To satisfy the strong law of action and reaction, $\vec{F}_{ij} = \nabla_j V_{ij} = -\nabla_i V_{ij} = -\vec{F}_{ji}$. Then, equation 2.22 can be written as,

$$\sum_i\sum_j \int_1^2 \vec{F}_{ij}d\vec{r}_{ij} = -\sum_i\sum_j \int_1^2 \nabla_i V_{ij}(d\vec{r}_i - d\vec{r}_j) = -\sum_i\sum_j \int_1^2 \nabla_{ij} V_{ij}d\vec{r}_{ij} \qquad (2.23)$$

Where $\vec{r}_{ij} = d\vec{r}_i - d\vec{r}_j$ and ∇_{ij} stands for the gradient with respect to $\vec{r}_{ij}$. The term $\nabla_{ij}V_{ij}d\vec{r}_{ij}$ in the equation 2.23 is for the ij pair of particles. The total work arising from internal forces then reduces to

$$\sum_i\sum_j \int_1^2 \vec{F}_{ij}d\vec{r}_{ij} = -\sum_i\sum_j \int_1^2 \frac{1}{2}\nabla_{ij} V_{ij}d\vec{r}_{ij} \qquad (2.24)$$

Combining the equation 2.24, equation 2.21 and equation 2.18, we see that,

$$\begin{aligned}
\int_1^2 W_{12}dW &= -\sum_i \int_1^2 \nabla_i V_i d\vec{r}_i - \sum_i\sum_j \int_1^2 \frac{1}{2}\nabla_{ij}V_{ij}d\vec{r}_{ij} \\
-V &= -\sum_i [(\nabla_i V_i)_2 - (\nabla_i V_i)_1] - \frac{1}{2}\sum_i\sum_j [(\nabla_{ij}V_{ij})_2 - (\nabla_{ij}V_{ij})_1] \\
V &= \sum_i \nabla_i V_i + \frac{1}{2}\sum_i\sum_j \nabla_{ij}V_{ij} \qquad (2.25)
\end{aligned}$$

[1] If $\vec{F}$ is conservative, $\nabla \times \vec{F} = 0$ then $\vec{F} = \nabla V$

The second term on the right in equation 2.25 will be called the internal potential energy of the system. In general, it need not be zero and, more important, it may vary as the system changes with time. Only for the particular class of systems known as rigid bodies the internal potential always be constant.

Reference books:

- Classical Mechanics - P V Panat
- Classical Mechanics, 3rd Ed - H.Goldstein, C.Poole and J.Safko

Chapter 3

The Lagrangean method

3.1 Constraints

Constraints means restrictions; constrained motion means restricted motion. Most of the motion that we encounter, is constrained motion. Most physical realizations of constrained motion involve surfaces of other bodies, for example,

1. **Motion of a billiard ball on the table:** Motion of a billiard ball is restricted by the boundaries of the table, and it moves on the surface of the table. If the centre of mass of a billiard ball of radius R moving on a billiard table of length $2a$ and breadth $2b$, must satisfy the relation

   $$-a + R \leq x \leq a - R, \qquad -b + R \leq y \leq b - R, \qquad z = R$$

   assuming that the origin of the coordinate axes is at the centre of the rectangular table and x and y axes are parallel to length and breadth respectively. i.e., a set of one equation and two inequalities, defines the motion of a billiard ball at all instants of time.

2. **The motion of a simple pendulum:** The bob of the pendulum moves in a vertical plane (say zx plane). Its distance from the fulcrum is fixed. Thus, if (x, y, z) is coordinate of the bob then,

   $$y = constant, \qquad z^2 + x^2 = l^2$$

   are the restrictions on the coordinates of the bob.

Physically, constrained motion is realised by the forces which arise when the object in motion is in contact with the constraining surfaces or curves. These forces, called constraint forces, are usually stiff elastic forces at the contact. If there are no constraints, motion of the particle is described by the trajectory $\vec{r}(t) = ix + jy + kz$ and by its momentum $\vec{p}(t) = ip_x + jp_y + kp_z$. Thus the position of the particle is specified by three coordinates. If there are N particles, *3N* independent coordinates are necessary for the position specification of the system at a time t. Presence of constraints may reduce the number of independent variables.

3.1.1 Classification of Constraints

1. (a) **Scleronomic:** constraint relations do not explicitly depend on time,
   (b) **Rheonomic:** constraint relations depend explicitly on time,

2. (a) **Holonomic:** conditions of constraint can be expressed as equations connecting the coordinates of the particles,
   (b) **Non holonomic:** constraint relations are not holonomic,

3. (a) **Conservative:** total mechanical energy of the system is conserved while performing, the constrained motion. Constraint forces do not do any work,
   (b) **Dissipative:** constraint forces do work and total mechanical energy is not conserved.

4. (a) **Bilateral:** at any point on the constraint surface both the forward and backward motions are possible. Constraint relations are not in the form of inequalities but are in the form of equations,
   (b) **Unilateral:** at some points no forward motion is possible. Constraint relations are expressed in the form of inequalities.

3.1.2 Holonomic and non holonomic constraints

If one can write the equations of constraints as

$$\begin{aligned} f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\ f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\ &. \\ &. \\ f_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\ f_{i+1}(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\ &. \\ &. \\ f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \end{aligned} \tag{3.1}$$

where $k < n$, then such constraints are known as holonomic constraints. The constraints which cannot be expressed in the form of algebraic equations are non holonomic constraints, however, they could be expressed as inequalities.

3.1.3 Examples of constraints

1. **Rigid body:** A rigid body is a system of particles such that the distance between any pair of particles remains constant in time. Thus the motion of a rigid body is constrained by the equations

$$\vec{r}_i - \vec{r}_k = const. \tag{3.2}$$

where the pair of subscripts $(i\ ,\ k)$ run over all distinct pairs of particles forming the body. Obviously this constraint is scleronomic. The constraint is also holonomic and bilateral. The constraint relations 3.2 can be written as

$$|\vec{r}_i - \vec{r}_k|^2 = const.$$

Taking differentials

$$(\vec{r}_i - \vec{r}_k).\Delta(\vec{r}_i - \vec{r}_k) = 0 \tag{3.3}$$

Work done by the system is

$$W = \sum_i \sum_k (\vec{F}_{ik}.\Delta\vec{r}_i + \vec{F}_{ki}.\Delta\vec{r}_k) \tag{3.4}$$

Let the internal force of constraint on the i^{th} particle due to the k^{th} particle be represented by $\vec{F}_{ik}$. By Newton's third law we have,

$$\vec{F}_{ik} = -\vec{F}_{ki} \tag{3.5}$$

Thus we have for the work done by $\vec{F}_{ik}$ due to a displacement $\Delta\vec{r}_i$ of the i^{th} particle,

$$\vec{F}_{ik}.\Delta\vec{r}_i = -\vec{F}_{ki}.\Delta\vec{r}_k \tag{3.6}$$

On combining equations 3.4 and 3.6 we can write the total work done by the system

$$W = \sum_i \sum_k \vec{F}_{ik}.(\Delta\vec{r}_i - \Delta\vec{r}_k) \tag{3.7}$$

Since all $\vec{F}_{ik}$ are the internal forces which arise purely due to interaction between all possible pairs of particles, it is only natural that $\vec{F}_{ik}$ will act parallel to the line joining the i^{th} and k^{th} particles. Thus we can write,

$$\vec{F}_{ik} = C_{ik}(\vec{r}_i - \vec{r}_k) \tag{3.8}$$

where C_{ik}'s are real constants and symmetric in i and k. Substituting in the above expression for the total work, we have

$$W = \sum_i \sum_k C_{ik}(\vec{r}_i - \vec{r}_k).(\Delta\vec{r}_i - \Delta\vec{r}_k) \tag{3.9}$$

In equation 3.9 each individual term of the summand is zero. Thus the constraint of rigidity is conservative in nature, apart from its being scleronomic, holonomic and bilateral.

2. **Deformable bodies:** Suppose that the deformation of the body is changing in time according to a certain prescribed function of time. Then the motion of such a body is constrained by the equation

$$|\vec{r}_i - \vec{r}_k| = f(t) \tag{3.10}$$

where $\vec{r}_i$ and $\vec{r}_k$ are position vectors and the pair of subscripts $(i\ ,\ k)$ runs over all distinct pairs of particles in the body. These constraint relations cannot give the total work $W = 0$. Hence this is a rheonomic, holonomic, bilateral and dissipative constraint.

3. **Gas in a spherical container of radius R**. Here if $\vec{r}_i$ is a position vector of the i^{th} gas molecule (origin is at the centre of the sphere) then

$$x_i^2 + y_i^2 + z_i^2 \leq R^2 \tag{3.11}$$

Thus, we have a constraint equation given by an inequality and hence is non holonomic constraint.

4. **Rolling without sliding:** Suppose a spherical ball is rolling on a plane without sliding. We assume that the surfaces in contact are perfectly rough. Thus the frictional forces are not negligible. Since the point of contact is not sliding, the frictional forces do not do any work, and hence the total mechanical energy of the rolling body is conserved. Thus the constraint is conservative. To obtain the constraint equation we note that rolling without sliding means that the relative velocity of the point of contact with respect to the plane is zero. Then the velocity v of any point P in the rolling body, as seen from a fixed frame of reference, is given by

$$v = V_{CM} + \vec{\omega} \times \vec{r} \tag{3.12}$$

where V_{CM} is the velocity of the centre of mass and $\vec{r}$ is measured from the CM to the point P under consideration. Thus the velocity of the point of contact is obtained by putting $\vec{r} = -r\hat{n}$ in equation 3.12, where $\hat{n}$ is the unit vector along the outward normal to the plane and r is the radius of the sphere. Since there is no sliding of this point we must have the instantaneous velocity v at the contact

$$v = V_{CM} - r(\vec{\omega} \times \hat{n}) = 0 \tag{3.13}$$

For a sphere this constraint is non integrable because ω is generally not expressible in the form of a total time derivative of any single coordinate. Thus the constraint is non holonomic. However, for a cylinder, $\omega = d\theta/dt$ where θ is the angle of rotation of the cylinder about its axis. Therefore this equation of constraint can be integrated and reduced to a holonomic form, giving a relation between r and the coordinates of the centre of mass.

3.2 Generalized coordinates

The problem of system of n particles can be solved when the number of constraint equations are less than *3n*. Let there be k equations of constraints $k < 3n$,

$$\begin{aligned}
f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
&. \\
&. \\
&. \\
&. \\
f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0
\end{aligned} \tag{3.14}$$

i.e., $3n - k$ coordinates may be regarded as free and which define the position of the system at any moment of time t. Then the number of independent coordinate to specify the motion at a given time t is $3n - k$. These independent coordinates are called *degrees of freedom*.

In the case of a free material particle, for instance, $n = 1$ and $k = 0$ so that it has $3n - k = 3$ degrees of freedom. If the particle is constrained to move on a surface whose equation may be taken as $f(x, y, z) = 0(z = 0)$, we clearly have $k = 1$ and therefore it would have $3n - k = 2$ degrees of freedom. On the other hand, for a dumb-bell shaped structure, with two particles connected by a rod of length l; a constraint equation becomes

$$f(x, y, z) = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0$$

we have $n = 2$ and $k = 1$, therefore it has $3n - k = 5$ degrees of freedom. The degrees of freedom are represented by $3n - k$ variables, $q_1, q_2, ..., q_{n-k}$. The old coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n$ are expressed in terms of $q's$ as,

$$\begin{aligned}
\vec{r}_1 &= \vec{r}_1(q_1, q_2, ..., q_{n-k}; t) \\
\vec{r}_2 &= \vec{r}_2(q_1, q_2, ..., q_{n-k}; t) \\
&. \\
&. \\
&. \\
&. \\
\vec{r}_n &= \vec{r}_n(q_1, q_2, ..., q_{n-k}; t)
\end{aligned} \tag{3.15}$$

3.2.1 Virtual displacement

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\vec{r}_i$, consistent with the forces and constraints imposed on the system at the given instant of time t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

3.2.2 Virtual work

Total work done by the external forces when virtual displacements are made in n particle system, is known as virtual work. If $\vec{F}_i^{(a)}$ be the applied force and $\vec{f}_i$ be the constraint force acting on i^{th} particle, the net force acting on the system is

$$\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i \tag{3.16}$$

If $\delta\vec{r}_i$ is the virtual displacement, the work done on the system is

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = \sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i \tag{3.17}$$

When system is in equilibrium

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = 0 \tag{3.18}$$

Thus equation 3.17 reduces to

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i = 0 \tag{3.19}$$

The virtual displacements $\delta\vec{r}_i$ are such that the constraint forces do no work ($\sum_i \vec{f}_i.\delta\vec{r}_i = 0$). Thus,

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i = 0 \tag{3.20}$$

i.e., The condition for static equilibrium is that the virtual work done by all the applied forces should vanish, provided the virtual work done by all the constraint forces vanishes. This is called the principle of virtual work.

3.3 D'Alembert's Principle

Consider the motion of n particle system. Then, by Newton's law,

$$\sum_i \vec{F}_i = \sum_i \dot{\vec{p}}_i \tag{3.21}$$

Combining equations 3.21 and 3.16,

$$\begin{aligned}
\sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i &= \sum_i \dot{\vec{p}}_i \\
\sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i - \sum_i \dot{\vec{p}}_i &= 0
\end{aligned} \tag{3.22}$$

The equation 3.22 states that the particles in the system will be in equilibrium under a force equal to the actual force plus a *reversed effective force* $-\dot{\vec{p}}_i$. The work done now can be written as

$$\sum_i (\vec{F}_i^{(a)} + \vec{f}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \tag{3.23}$$

The virtual displacements $\delta\vec{r}_i$ are such that the constraint forces do no work ($\sum_i \vec{f}_i.\delta\vec{r}_i = 0$). Thus,

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \tag{3.24}$$

The equation 3.24 is called D'Alembert's Principle. D'Alembert's principle does not involve forces of constraint. i.e., any dynamical problem could be converted into an effective static problem.

3.4 Lagrange's Equations of the second kind

In D'Alembert's principle, the virtual displacements $\delta\vec{r}_i$ are not independent. Therefore, the D'Alembert's equation is a single equation. If the constraints are holonomic, we use independent set of variables $\{q_i\}$. When this is done, we get, n equations, one each for each q. These equations are Lagrange's equations. For n particle system the D'Alembert's equation is,

$$\sum_i \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right).\delta\vec{r}_i = 0 \tag{3.25}$$

If we have n particles at $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n$ and there are k equations of holonomic constraints, then there are $3n - k = m$ generalized coordinates. They are denoted by $q_1, q_2, ..., q_m$. Thus we have,

$$\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_m; t) = \vec{r}_i(q_j, t) \tag{3.26}$$

Velocity of the i^{th} particle, v_i is given as (on differentiating the equation 3.26),

$$\frac{d\vec{r}_i}{dt} = \sum_{j=1}^{m} \frac{\delta\vec{r}_i}{\delta q_j}\frac{\delta q_j}{\delta t} + \frac{\delta\vec{r}_i}{\delta t}$$

$$v_i = \sum_{j=1}^{m} \frac{\delta\vec{r}_i}{\delta q_j}\dot{q}_j + \frac{\delta\vec{r}_i}{\delta t} \tag{3.27}$$

$$\frac{\delta v_i}{\delta\dot{q}_j} = \frac{\delta\vec{r}_i}{\delta q_j} \tag{3.28}$$

The virtual work of the system is

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = \sum_i\sum_j \vec{F}_i.\frac{\delta\vec{r}_i}{\delta q_j}\delta q_j \qquad \text{(by using equation ??)}$$

$$\sum_i \vec{F}_i.\delta\vec{r}_i = \sum_j Q_j\delta q_j \tag{3.29}$$

where the Q_j are called the components of the generalized force, defined as

$$Q_j = \sum_i \vec{F}_i.\frac{\delta\vec{r}_i}{\delta q_j} \tag{3.30}$$

Note that $q's$ need not have the dimensions of length, so the $Q's$ do not necessarily have the dimensions of force, but $Q_j\delta q_j$ must always have the dimensions of work. For example, Q_j might be a torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j d\theta_j$ a differential of work.
Again consider the equation 3.29,

$$\sum_j Q_j\delta q_j = \sum_i \vec{F}_i.\delta\vec{r}_i$$

$$= \sum_i \dot{\vec{p}}_i.\delta\vec{r}_i$$

$$= \sum_i m_i\ddot{\vec{r}}_i.\delta\vec{r}_i$$

$$= \sum_i\sum_j m_i\ddot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\delta q_j$$

$$\sum_j Q_j\delta q_j = \sum_i\sum_j \left[\frac{d}{dt}\left(m_i\dot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\right) - m_i\dot{\vec{r}}_i.\frac{d}{dt}\left(\frac{\delta\vec{r}_i}{\delta q_j}\right)\right]\delta q_j \tag{3.31}$$

We can see from the equation 3.31, $\vec{r_i}$ is differentiable with respect to both t and q_j, we can interchange the differentiation with respect to t and q_j in equation 3.31.

$$\begin{aligned}\sum_j Q_j\delta q_j &= \sum_i\sum_j\left[\frac{d}{dt}\left(m_i\dot{\vec{r_i}}.\frac{\delta\vec{r_i}}{\delta q_j}\right)-m_i\dot{\vec{r_i}}.\frac{\delta}{\delta q_j}\left(\frac{d\vec{r_i}}{dt}\right)\right]\delta q_j\\ &= \sum_i\sum_j\left[\frac{d}{dt}\left(m_i\dot{\vec{r_i}}.\frac{\delta\vec{r_i}}{\delta q_j}\right)-m_i\dot{\vec{r_i}}.\frac{\delta\vec{v_i}}{\delta q_j}\right]\delta q_j \qquad (3.32)\end{aligned}$$

From the equation 3.28, we can write

$$\frac{\delta v_i}{\delta\dot{q_j}} = \frac{\delta\vec{r_i}}{\delta q_j} \qquad (3.33)$$

Equation 3.33 in equation 3.32 gives,

$$\begin{aligned}\sum_j Q_j\delta q_j &= \sum_i\sum_j\left[\frac{d}{dt}\left(m_i\vec{v_i}.\frac{\delta v_i}{\delta\dot{q_j}}\right)-m_i\vec{v_i}.\frac{\delta\vec{v_i}}{\delta q_j}\right]\delta q_j\\ &= \sum_j\left\{\frac{d}{dt}\left[\frac{\delta}{\delta\dot{q_j}}\left(\sum_i\frac{1}{2}m_i\vec{v_i}^{\,2}\right)\right]-\frac{\delta}{\delta q_j}\left(\sum_i\frac{1}{2}m_i\vec{v_i}^{\,2}\right)\right\}\delta q_j\\ \sum_j Q_j\delta q_j &= \sum_j\left[\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{q_j}}\right)-\frac{\delta T}{\delta q_j}\right]\delta q_j\end{aligned}$$

where $T=\sum_i\frac{1}{2}m_i\vec{v_i}^{\,2}$ is the kinetic energy of the system.

$$\sum_j\left[\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{q_j}}\right)-\frac{\delta T}{\delta q_j}-Q_j\right]\delta q_j = 0$$

Since all δq_j are independent whereas, $\delta\vec{r_i}$ were not. Thus,

$$\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{q_j}}\right)-\frac{\delta T}{\delta q_j}-Q_j = 0 \qquad (3.34)$$

When the forces F_i are derivable from a scalar potential function V

$$F_i = -\nabla_i V \qquad (3.35)$$

Equation 3.35 in 3.30 gives

$$\begin{aligned}Q_j &= -\nabla_i V.\frac{\delta\vec{r_i}}{\delta q_j}\\ Q_j &= -\left(i\frac{\delta V}{\delta x_i}+j\frac{\delta V}{\delta y_i}+k\frac{\delta V}{\delta z_i}\right).\left(i\frac{\delta x_i}{\delta q_j}+j\frac{\delta y_i}{\delta q_j}+k\frac{\delta z_i}{\delta q_j}\right) \qquad (3.36)\end{aligned}$$

The equation 3.36 is exactly the same expression for the partial derives of a function $-V(r_l, r_2, ..., r_n; t)$ with respect to q_j.

$$Q_j = -\frac{\delta V}{\delta q_j} \qquad (3.37)$$

On combining the equations 3.34 and 3.37

$$\begin{aligned}\frac{d}{dt}\left(\frac{\delta T}{\delta\dot{q_j}}\right)-\frac{\delta T}{\delta q_j}+\frac{\delta V}{\delta q_j} &= 0\\ \frac{d}{dt}\left(\frac{\delta T}{\delta\dot{q_j}}\right)-\frac{\delta(T-V)}{\delta q_j} &= 0 \qquad (3.38)\end{aligned}$$

The equations of motion in the form 3.38 are restricted to conservative systems, only when V is independent of time. Hence the potential V is a function of generalized coordinates q_j only, and does not depend upon $\dot{q}_j$. We define a new function, the Lagrangian L, as

$$L(q_j,\dot{q_j}) = T(q_j,\dot{q_j})-V(q_j) \qquad (3.39)$$

The partial derives of the equation 3.39 are

$$\frac{\delta L}{\delta\dot{q_j}} = \frac{\delta T}{\delta\dot{q_j}} \qquad (3.40)$$

$$\frac{\delta L}{\delta q_j} = \frac{\delta T}{\delta q_j}-\frac{\delta V}{\delta q_j}=\frac{\delta(T-V)}{\delta q_j} \qquad (3.41)$$

Equations 3.40 and 3.41 in equation 3.38,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_j}\right) - \frac{\delta L}{\delta q_j} = 0, \qquad j = 1, 2, 3, ..., m. \tag{3.42}$$

Equation 3.42 gives set of m equations. These m equations, one for each independent generalized coordinates, are known as Lagrange's Equations of motion of the second kind in a potential field.

3.4.1 Simple applications of the Lagrangian formulation

Kinetic energy T is given by

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i v_i^2 \\ &= \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{m} \frac{\delta \vec{r}_i}{\delta q_j}\dot{q}_j + \frac{\delta \vec{r}_i}{\delta t}\right)^2 \qquad \text{by using equation 3.28} \\ &= \frac{1}{2}\sum_i m_i \left[\sum_{j=1}^{m} \frac{\delta \vec{r}_i}{\delta q_j}\dot{q}_j \sum_{k=1}^{m} \frac{\delta \vec{r}_i}{\delta q_k}\dot{q}_k + 2\sum_{j=1}^{m} \frac{\delta \vec{r}_i}{\delta q_j}\dot{q}_j \frac{\delta \vec{r}_i}{\delta t} + \left(\frac{\delta \vec{r}_i}{\delta t}\right)^2\right] \\ &= \frac{1}{2}\sum_j\sum_k\sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta q_k}\dot{q}_j\dot{q}_k + \sum_j\sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta t}\dot{q}_j + \frac{1}{2}\sum_i m_i \left(\frac{\delta \vec{r}_i}{\delta t}\right)^2 \\ T &= \frac{1}{2}\sum_j\sum_k m_{jk}\dot{q}_j\dot{q}_k + \sum_j m_j\dot{q}_j + m_0 \end{aligned} \tag{3.43}$$

where

$$m_{jk} = \sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta q_k} \tag{3.44}$$

$$m_j = \sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta t}$$

$$m_0 = \frac{1}{2}\sum_i m_i \left(\frac{\delta \vec{r}_i}{\delta t}\right)^2$$

Thus, the kinetic energy of a system can always be written as the sum of three homogeneous functions of the generalized velocities,

$$T = T_2 + T_1 + T_0 \tag{3.45}$$

where T_0 is independent of the generalized velocities, T, is linear in the velocities, and T_2 is quadratic in the velocities. If the transformation equations do not contain the time explicitly, as may occur when the constraints are independent of time (scleronomous),

$$m_0 = 0 \text{ and } m_j = 0 \quad \Longrightarrow \quad T_0 = 0 \text{ and } T_1 = 0.$$

Then

$$T = T_2 = \frac{1}{2}\sum_j\sum_k m_{jk}\dot{q}_j\dot{q}_k \tag{3.46}$$

Thus, T is always a homogeneous quadratic form in the generalized velocities.

1. **Motion of a single particle**

   (a) *Using Cartesian coordinates:* If (x, y, z) are the Cartesian coordinates at time t of a free material point of mass m moving in a potential field $V(x, y, z)$, we may take $q_1 = x, q_2 = y, q_3 = z$ as there are no equations of constraint. The applied force $\vec{F}$ on the particle has the components $-\frac{\delta V}{\delta x}, -\frac{\delta V}{\delta y}, -\frac{\delta V}{\delta z}$, while the kinetic energy T is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus the Lagrangian for the particle is

$$T - V = L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \tag{3.47}$$

$$\frac{\delta L}{\delta \dot{x}} = m\dot{x} \tag{3.48}$$

$$\frac{\delta L}{\delta x} = -\frac{\delta V}{\delta x} \tag{3.49}$$

Equations 3.48 and 3.49 in equation 3.42,

$$\frac{d}{dt}(m\dot{x}) + \frac{\delta V}{\delta x} = 0$$

$$m\ddot{x} = -\frac{\delta V}{\delta x} = F_x$$

$$\text{Similarly} \quad m\ddot{y} = F_y, \qquad m\ddot{z} = F_z$$

The Lagrangian equations of motion are

$$m\ddot{x} = F_x, \qquad m\ddot{y} = F_y, \qquad m\ddot{z} = F_z \tag{3.50}$$

Equation 3.50 gives Newton's equations of motion.

(b) *Using cylindrical polar coordinates*: If (r, θ, z) are the cylindrical coordinates at time t of a free material

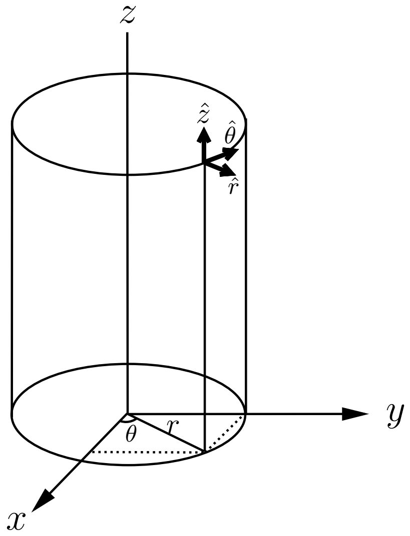

Figure 3.1: Cylindrical coordinates; $d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + dz\,\hat{k}$.

point of mass m, we may take $(q_1 = r, q_2 = \theta, q_3 = z)$ as there are no equations of constraint. The applied force $\vec{F}$ on the particle has the components in generalized coordinates (Q_r, Q_θ, Q_z).
The three Lagrangian equations can be written by using the equation 3.34,

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{r}}\right) - \frac{\delta T}{\delta r} - Q_r = 0 \tag{3.51}$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\theta}}\right) - \frac{\delta T}{\delta \theta} - Q_\theta = 0 \tag{3.52}$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{z}}\right) - \frac{\delta T}{\delta z} - Q_z = 0 \tag{3.53}$$

The position vector $\vec{r}$ in cylindrical coordinates,

$$\vec{r} = r\hat{r} + r\theta\hat{\theta} + z\hat{k}$$

where $\hat{r}, \hat{\theta}$ and $\hat{k}$ are unit vectors in the r, θ and z directions, respectively. The components of the force in generalized coordinates can be obtained from the equation 3.30 as,

$$Q_r = F_r\frac{\delta \vec{r}}{\delta r} = F_r\hat{r} \tag{3.54}$$

$$Q_\theta = F_\theta\frac{\delta \vec{r}}{\delta \theta} = F_\theta r\hat{\theta} \tag{3.55}$$

$$Q_z = F_z\frac{\delta \vec{r}}{\delta z} = F_z\hat{k} \tag{3.56}$$

In cylindrical coordinates

$$x = rcos\theta, \qquad y = rsin\theta, \qquad z = z$$

$$\frac{dx}{dt} = -rsin\theta\,\frac{d\theta}{dt} + \frac{dr}{dt}cos\theta, \qquad \frac{dy}{dt} = rcos\theta\,\frac{d\theta}{dt} + \frac{dr}{dt}sin\theta, \qquad \frac{dz}{dt} = \frac{dz}{dt}$$

$$\dot{x} = -rsin\theta\,\dot{\theta} + \dot{r}cos\theta, \qquad \dot{y} = rcos\theta\,\dot{\theta} + \dot{r}sin\theta, \qquad \dot{z} = \dot{z}$$

The kinetic energy T is

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\
&= \frac{1}{2}m(-rsin\theta\ \dot{\theta} + \dot{r}cos\theta)^2 + (rcos\theta\ \dot{\theta} + \dot{r}sin\theta)^2 + \dot{z}^2 \\
T &= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2] \qquad (3.57)\\
\frac{\delta T}{\delta \dot{r}} &= m\dot{r}, \qquad \frac{\delta T}{\delta r} = mr\dot{\theta}^2 \qquad (3.58)\\
\frac{\delta T}{\delta \dot{\theta}} &= mr^2\dot{\theta}, \qquad \frac{\delta T}{\delta \theta} = 0 \qquad (3.59)\\
\frac{\delta T}{\delta \dot{z}} &= m\dot{z}, \qquad \frac{\delta T}{\delta z} = 0 \qquad (3.60)
\end{aligned}
$$

Equations 3.54 and 3.58 in equation 3.51,

$$
\begin{aligned}
\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - F_r &= 0 \\
m\ddot{r} - mr\dot{\theta}^2 &= F_r \qquad (3.61)
\end{aligned}
$$

If r is constant, $F_r = -mr\dot{\theta}^2$ being the centripetal acceleration.
Equations 3.55 and 3.59 in equation 3.52,

$$
\begin{aligned}
\frac{d}{dt}(mr^2\dot{\theta}) &- rF_\theta = 0 \\
\frac{d\vec{L}}{dt} &= rF_\theta = N^l \qquad (3.62)
\end{aligned}
$$

where $\vec{L} = mr^2\dot{\theta}$ is the angular momentum and N^l is the applied torque.
Equations 3.56 and 3.60 in equation 3.53,

$$
\begin{aligned}
\frac{d}{dt}(m\dot{z}) &- F_z = 0 \\
m\ddot{z} &= F_z \qquad (3.63)
\end{aligned}
$$

(c) *Using spherical polar coordinates*: If (r, θ, ϕ) are the spherical polar coordinates at time t of a free material point of mass m, we may take $(q_1 = r, q_2 = \theta, q_3 = \phi)$ as there are no equations of constraint. The applied force $\vec{F}$ on the particle has the components in generalized coordinates (Q_r, Q_θ, Q_ϕ).
The three Lagrangian equations can be written by using the equation 3.34,

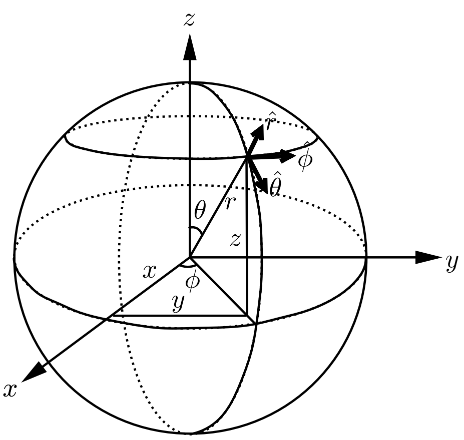

Figure 3.2: Spherical coordinates; $d\vec{r} = dr\,\hat{r} + rsin\theta d\phi\,\hat{\phi} + rd\theta\,\hat{\theta}$.

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{r}}\right) - \frac{\delta T}{\delta r} - Q_r &= 0 \qquad (3.64)\\
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\phi}}\right) - \frac{\delta T}{\delta \phi} - Q_\phi &= 0 \qquad (3.65)\\
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\theta}}\right) - \frac{\delta T}{\delta \theta} - Q_\theta &= 0 \qquad (3.66)
\end{aligned}
$$

The position vector $\vec{r}$ in spherical coordinates,

$$\vec{r} = r\,\hat{r} + rsin\theta\phi\,\hat{\phi} + r\theta\,\hat{\theta}$$

where $\hat{r}, \hat{\phi}$ and $\hat{\theta}$ are unit vectors in the r, ϕ and θ directions, respectively. The components of the force in generalized coordinates can be obtained from the equation 3.30 as,

$$Q_r = F_r\frac{\delta\vec{r}}{\delta r} = F_r\ \hat{r} \tag{3.67}$$

$$Q_\phi = F_\phi\frac{\delta\vec{r}}{\delta\phi} = F_\phi rsin\theta\ \hat{\phi} \tag{3.68}$$

$$Q_\theta = F_\theta\frac{\delta\vec{r}}{\delta\theta} = F_\theta r\ \hat{\theta} \tag{3.69}$$

In spherical coordinates

$$\begin{aligned} x &= rcos\phi\ sin\theta, \qquad y = rsin\phi\ sin\theta, \qquad z = rcos\theta \\ \dot{x} &= \dot{r}cos\phi\ sin\theta - rsin\phi\ \dot{\phi}\ sin\theta + rcos\phi\ cos\theta\ \dot{\theta} \\ \dot{y} &= \dot{r}sin\phi\ sin\theta + rcos\phi\ \dot{\phi}\ sin\theta + rsin\phi\ cos\theta\ \dot{\theta} \\ \dot{z} &= \dot{r}cos\theta - rsin\theta\ \dot{\theta} \end{aligned}$$

The kinetic energy T is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$T = \frac{1}{2}m\left[\dot{r}^2 + \left(rsin\theta\dot{\phi}\right)^2 + \left(r\dot{\theta}\right)^2\right] \tag{3.70}$$

$$\frac{\delta T}{\delta\dot{r}} = m\dot{r}, \qquad \frac{\delta T}{\delta r} = m\left(rsin^2\theta\dot{\phi}^2 + r\dot{\theta}^2\right) \tag{3.71}$$

$$\frac{\delta T}{\delta\dot{\phi}} = mr^2sin^2\theta\ \dot{\phi}, \qquad \frac{\delta T}{\delta\phi} = 0 \tag{3.72}$$

$$\frac{\delta T}{\delta\dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{\delta T}{\delta\theta} = mr^2\dot{\phi}^2cos\theta \tag{3.73}$$

Equations 3.67 and 3.71 in equation 3.64,

$$\frac{d}{dt}(m\dot{r}) - m\left(rsin^2\theta\dot{\phi}^2 + r\dot{\theta}^2\right) - F_r = 0$$

$$m\ddot{r} - m\left(rsin^2\theta\dot{\phi}^2 + r\dot{\theta}^2\right) = F_r \tag{3.74}$$

Equations 3.68 and 3.72 in equation 3.65,

$$\frac{d}{dt}(mr^2sin^2\theta\dot{\phi}) - rsin\theta\ F_\phi = 0$$

$$\frac{d}{dt}(mr^2sin^2\theta\dot{\phi}) = rsin\theta\ F_\phi \tag{3.75}$$

Equations 3.69 and 3.73 in equation 3.66,

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\dot{\phi}^2cos\theta - rF_\theta = 0$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - mr^2\dot{\phi}^2cos\theta = rF_\theta \tag{3.76}$$

2. **Atwood's machine:** Figure 3.3 shows the schematic diagram of Atwood's machine which is an example of a conservative system with holonomic, scleronomous constraint (the pulley is assumed frictionless and massless). Clearly there is only one independent coordinate y, the position of the other weight being determined by the constraint that the length of the rope between them is l. The Lagrangian equation for the motion can be written by using equation3.42 as,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{y}}\right) - \frac{\delta L}{\delta y} = 0 \tag{3.77}$$

The potential energy is

$$V = -M_1gy - M_2g(l - y) \tag{3.78}$$

The kinetic energy of the system is

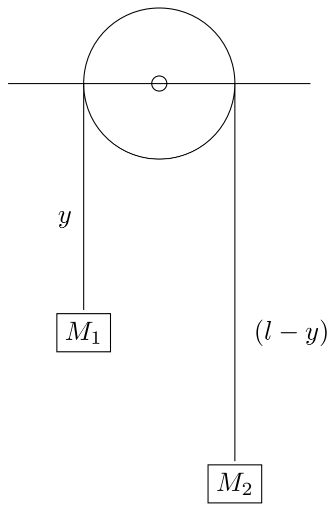

Figure 3.3: Atwood's machine.

$$T = \frac{1}{2}(M_1 + M_2)\dot{y}^2 \tag{3.79}$$

The Lagrangian L is

$$L = T - V = \frac{1}{2}(M_1 + M_2)\dot{y}^2 + M_1 gy + M_2 g(l - y) \tag{3.80}$$

$$L\frac{\delta L}{\delta \dot{y}} = (M_1 + M_2)\dot{y} \tag{3.81}$$

$$L\frac{\delta L}{\delta y} = M_1 g - M_2 g \tag{3.82}$$

Equations 3.81 and 3.82 in equation 3.77,

$$\begin{aligned} \frac{d}{dt}\left[(M_1 + M_2)\dot{y}\right] - M_1 g + M_2 g &= 0 \\ (M_1 + M_2)\ddot{y} &= (M_1 - M_2)g \\ \ddot{y} &= \frac{(M_1 - M_2)}{(M_1 + M_2)}g \end{aligned}$$

This is the familiar result obtained by more elementary means. This emphasizes that the forces of constraint here the tension in the rope appear nowhere in the Lagrangian formulation. Also the tension in the rope can not be found directly by the Lagrangian method.

3. **A bead (or ring) sliding on a uniformly rotating wire in a force-free space:** Consider beads in a straight wire, and is rotated uniformly with angular velocity $\vec{\omega}$ about some fixed axis perpendicular to the wire. This example has been chosen as a simple illustration of a constraint being time dependent, with the rotation axis along z and the wire in the xy plane. Beads can move along wire, only one Lagrangian equation as using the equation 3.42,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{r}}\right) - \frac{\delta L}{\delta r} = 0 \tag{3.83}$$

The transformation equations explicitly contain the time,

$$\begin{aligned} x &= rcos\omega t, & y &= rsin\omega t \\ \dot{x} &= -rsin\omega t\ \omega + \dot{r}cos\omega t, & \dot{y} &= rcos\omega t\ \omega + \dot{r}sin\omega t, \end{aligned}$$

The kinetic energy T is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$T = \frac{1}{2}m[\dot{r}^2 + (r\omega)^2] \tag{3.84}$$

The Lagrangian L is

$$\begin{aligned} T - V = L &= \frac{1}{2}m[\dot{r}^2 + (r\omega)^2] \\ \frac{\delta L}{\delta \dot{r}} &= m\dot{r} \qquad (3.85) \\ \frac{\delta L}{\delta r} &= mr\omega^2 \qquad (3.86) \end{aligned}$$

Equations 3.85 and 3.86 in equation 3.83,

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) - mr\omega^2 &= 0 \\ \ddot{r} &= r\omega^2 \qquad (3.87) \end{aligned}$$

The equation 3.87 is the familiar simple harmonic oscillator equation with a change of sign. The solution of the equation is $r = e^{\omega t}$ shows that the bead moves exponentially outward because of the centripetal acceleration. But the method cannot furnish the force of constraint that keeps the bead on the wire.

3.5 Velocity dependent potential

Consider an electric charge, q, of mass m moving at a velocity, v, in an electric field, $\vec{E}$, and a magnetic field, $\vec{B}$, which may depend upon time and position. The electric charge experiences a force, called the Lorentz force, given by

$$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] \qquad (3.88)$$

Both $\vec{E}(t, x, y, z)$ and $\vec{B}(t, x, y, z)$ are continuous functions of time and position derivable from a scalar potential $\phi(t, x, y, z)$ and a vector potential $\vec{A}(t, x, y, z)$.
Faraday's law of electromagnetic induction is

$$\begin{aligned} \nabla \times \vec{E} &= -\frac{\delta \vec{B}}{\delta t} = -\frac{\delta}{\delta t}(\nabla \times \vec{A}) = -\nabla \times \frac{\delta \vec{A}}{\delta t} \\ \nabla \times \vec{E} + \nabla \times \frac{\delta \vec{A}}{\delta t} &= 0 \\ \nabla \times \left(\vec{E} + \frac{\delta \vec{A}}{\delta t} \right) &= 0 \end{aligned}$$

Thus, $\vec{E} + \delta\vec{A}/\delta t$ is gradient of some scalar function ϕ. i.e.,

$$\begin{aligned} \left(\vec{E} + \frac{\delta \vec{A}}{\delta t} \right) &= -\nabla\phi \\ \vec{E} &= -\nabla\phi - \frac{\delta \vec{A}}{\delta t} \qquad (3.89) \end{aligned}$$

Equation 3.89 in 3.88,

$$\vec{F} = q\left[-\nabla\phi - \frac{\delta \vec{A}}{\delta t} + (\vec{v} \times \nabla \times \vec{A}) \right] \qquad (3.90)$$

Taking x component of $\vec{F}$,

$$\begin{aligned} F_x &= q\left[-\frac{\delta \phi}{\delta x} - \frac{\delta A_x}{\delta t} + (\vec{v} \times \nabla \times \vec{A})_x \right] \\ &= q\left[-\frac{\delta \phi}{\delta x} - \frac{\delta A_x}{\delta t} + v_y\,(\nabla \times A)_z - v_z\,(\nabla \times A)_y \right] \\ &= q\left[-\frac{\delta \phi}{\delta x} - \frac{\delta A_x}{\delta t} + v_y\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y} \right) + v_z\left(\frac{\delta A_z}{\delta x} - \frac{\delta A_x}{\delta z} \right) \right] \qquad (3.91) \end{aligned}$$

We can write

$$\begin{aligned} \frac{dA_x}{dt} &= \frac{\delta A_x}{\delta t} + \frac{\delta A_x}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta A_x}{\delta y}\frac{\delta y}{\delta t} + \frac{\delta A_x}{\delta z}\frac{\delta z}{\delta t} \\ -\frac{\delta A_x}{\delta t} &= -\frac{dA_x}{dt} + \frac{\delta A_x}{\delta x}v_x + \frac{\delta A_x}{\delta y}v_y + \frac{\delta A_x}{\delta z}v_z \qquad (3.92) \end{aligned}$$

Equation 3.92 in 3.91,

$$\begin{aligned}
F_x &= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta A_x}{\delta x}v_x + \frac{\delta A_x}{\delta y}v_y + \frac{\delta A_x}{\delta z}v_z + v_y\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right) + v_z\left(\frac{\delta A_z}{\delta x} - \frac{\delta A_x}{\delta z}\right)\right] \\
&= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta A_x}{\delta x}v_x + \frac{\delta A_y}{\delta x}v_y + \frac{\delta A_z}{\delta x}v_z\right] \\
&= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta}{\delta x}(A_x v_x + A_y v_y + A_z v_z)\right] \\
F_x &= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta}{\delta x}(\vec{A}.\vec{v})\right] \\
F_x &= q\left[-\frac{\delta}{\delta x}(\phi - \vec{A}.\vec{v}) - \frac{dA_x}{dt}\right] \qquad (3.93)
\end{aligned}$$

We also can write

$$\begin{aligned}
-\frac{dA_x}{dt} &= -\frac{d}{dt}\frac{\delta}{\delta v_x}(\vec{A}.\vec{v}) \\
-\frac{dA_x}{dt} &= \frac{d}{dt}\frac{\delta}{\delta v_x}(\phi - \vec{A}.\vec{v}) \qquad (3.94)
\end{aligned}$$

Equation 3.94 in 3.93,

$$\begin{aligned}
F_x &= q\left[-\frac{\delta}{\delta x}(\phi - \vec{A}.\vec{v}) + \frac{d}{dt}\frac{\delta}{\delta v_x}(\phi - \vec{A}.\vec{v})\right] \\
F_x &= -\frac{\delta U}{\delta x} + \frac{d}{dt}\frac{\delta U}{\delta v_x} \qquad (3.95)
\end{aligned}$$

where $U = q(\phi - \vec{A}.\vec{v})$ is the velocity dependent potential. The Lagrange's equation was written as (equation 3.34),

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j} &= Q_j \\
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{x}}\right) - \frac{\delta T}{\delta x} &= Q_x = F_x \qquad (3.96)
\end{aligned}$$

On combining equations 3.95 and 3.96,

$$\begin{aligned}
\frac{d}{dt}\frac{\delta}{\delta \dot{x}}(T-U) - \frac{\delta}{\delta x}(T-U) &= 0 \\
\frac{d}{dt}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} &= 0 \qquad (3.97)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{d}{dt}\frac{\delta L}{\delta \dot{y}} - \frac{\delta L}{\delta y} &= 0 \qquad (3.98) \\
\frac{d}{dt}\frac{\delta L}{\delta \dot{z}} - \frac{\delta L}{\delta z} &= 0 \qquad (3.99)
\end{aligned}$$

where $L = T - U = \frac{1}{2}mv^2 - q(\phi - \vec{A}.\vec{v})$ is the Lagrangian for a charged particle in electromagnetic field. Thus, even electromagnetic forces can be accommodated in Lagrange's formulation.

3.6 Hamilton's principle

The motion of a conservative system from its configuration at time t_1 to its configuration at time t_2 is such that the line integral between the time t_1 and t_2 of the Lagrangian of the system has a stationary value for the actual path of the motion.

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \qquad (3.100)$$

$L = T - V$ is the Lagrangian. Since $\int Ldt$ has the dimensions of *energy* $\times$ *time* called action, the principle is sometimes referred to as the principle of least action. The integral is called the action integral.
Then the variation of the action integral for fixed time t_1 and t_2 must be zero. i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0 \qquad (3.101)$$

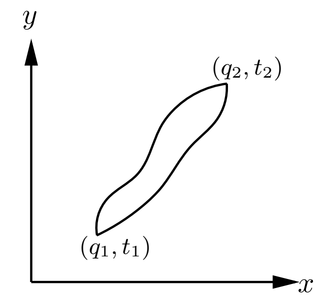

Figure 3.4: Varied paths of the function of $q(t)$ in the one-dimensional extremum problem.

I has a stationary value relative to paths differing infinitesimally from the correct function $q(t)$. Let $\eta(t)$ be a continuous function with continuous first derivative and $\eta(t_1) = \eta(t_2) = 0$. We construct an another curve $q(t, \alpha)$ as

$$q(t,\alpha) = q(t,0) + \alpha\eta(t) \tag{3.102}$$

Then, with different values of α we will get different paths. For $\alpha = 0$, equation 3.102 gives the curve $q(t, 0) = q(t)$. For simplicity, it is assumed that both the correct path $q(t)$ and the auxiliary function $\eta(t)$ are well-behaved functions and are continuous and nonsingular between t_1 and t_2, with continuous first and second derivatives in the same interval. For any such parametric family of curves, equation 3.100 can be written as,

$$I(\alpha) = \int_{t_1}^{t_2} L\{q(t,\alpha), \dot{q}(t,\alpha), t\}\ dt \tag{3.103}$$

and the condition for obtaining a stationary point is,

$$\left(\frac{dI}{d\alpha}\right)_{\alpha=0} = 0$$

Consider the equation 3.100,

$$\begin{aligned}
I &= \int_{t_1}^{t_2} L(q,\dot{q},t)\ dt \\
\frac{\delta I}{\delta\alpha}d\alpha &= \int_{t_1}^{t_2}\left(\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha + \frac{\delta L}{\delta\dot{q}}\frac{\delta\dot{q}}{\delta\alpha}d\alpha\right)\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \int_{t_1}^{t_2}\frac{\delta L}{\delta\dot{q}}\frac{\delta\dot{q}}{\delta\alpha}d\alpha\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \int_{t_1}^{t_2}\frac{\delta L}{\delta\dot{q}}\frac{\delta^2 q}{\delta\alpha\delta t}d\alpha\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \int_{t_1}^{t_2}\frac{\delta L}{\delta\dot{q}}\frac{\delta}{\delta t}\left(\frac{\delta q}{\delta\alpha}\right)d\alpha\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \left.\frac{\delta L}{\delta\dot{q}}\frac{\delta q}{\delta\alpha}\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\delta q}{\delta\alpha}\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{q}}\right)d\alpha\ dt
\end{aligned}$$

Since $\left(\frac{\delta q}{\delta\alpha}\right)_{t_1} = \left(\frac{\delta q}{\delta\alpha}\right)_{t_2} = 0,$

$$\begin{aligned}
\frac{\delta I}{\delta\alpha}d\alpha &= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt - \int_{t_1}^{t_2}\frac{\delta q}{\delta\alpha}\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{q}}\right)d\alpha\ dt \\
\frac{\delta I}{\delta\alpha}d\alpha &= \int_{t_1}^{t_2}\left(\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta\dot{q}}\right)\frac{\delta q}{\delta\alpha}d\alpha\ dt
\end{aligned}$$

At $\alpha = 0, \left(\frac{\delta q}{\delta\alpha}\right)_0 d\alpha = \delta q$ and $\left(\frac{\delta I}{\delta\alpha}\right)_0 d\alpha = \delta I$

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta\dot{q}}\right)\delta q\ dt$$

According to Hamilton's principle $\delta I = 0$,

$$\int_{t_1}^{t_2} \left(\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} \right) \delta q \; dt \quad = \quad 0 \tag{3.104}$$

Since the q variables are independent, the variations δq are independent. In the equation 3.104, the coefficients of δq must vanish separately.

$$\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} \quad = \quad 0 \tag{3.105}$$

The equation 3.105 is the Lagrange equation of motion. This equation is valid for any function $f(q, \dot{q}, t)$ and is called the Euler-Lagrange differential equation.

Reference books:

- Classical Mechanics - P V Panat
- Classical Mechanics, 3rd Ed - H.Goldstein, C.Poole and J.Safko
- Classical Mechanics - N C Rana and P S Joag

Chapter 4

Rigid body dynamics

A rigid body is defined as a system of mass points subject to the holonomic constraints that the distances between all pairs of points remain constant throughout the motion.

4.1 The independent coordinates of a rigid body - Degrees of freedom

A rigid body with N particles can at most have $3N$ degrees of freedom, but these are greatly reduced by the constraints, which can be expressed as equations of the form

$$r_{ij} = c_{ij} \tag{4.1}$$

where r_{ij} is distance between any i^{th} and j^{th} particle and c_{ij} is constant. But all these relations are not independent. Consider three non collinear points *1, 2* and *3* of rigid body, as shown in the figure 4.1. If there is no constraint equations, the number of degrees of freedom becomes 9. If we write the constraint equations,

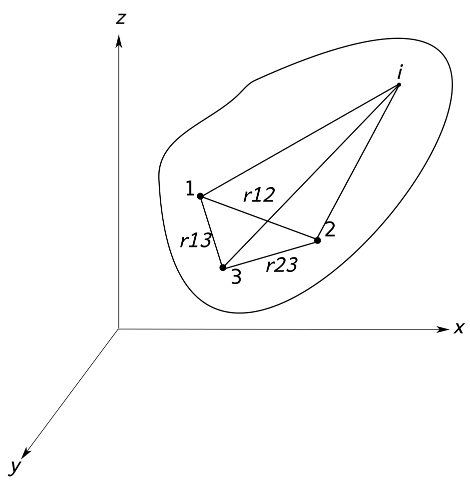

Figure 4.1: Rigid body with three non collinear points.

$$r_{12} = c_{12}, \qquad r_{23} = c_{23} \qquad r_{31} = c_{31} \tag{4.2}$$

Then the number of degrees of freedom becomes $9 - 3 = 6$. That is only six coordinates are needed. Of these 6 coordinates, 3 are the coordinates of CM of rigid body with respect to the fixed axis and the other three generalized coordinates are the angles that the axis of rotation makes with the fixed coordinate system. A rigid body in space thus needs six independent generalized coordinates to specify its configuration, no matter how many particles it may contain.

4.2 Rotation about an axis, Orthogonal matrix

Consider a rigid body in which Cartesian coordinate system x', y', z' fixed at the point O'. $\vec{r}'$ is the position vector of a mass point rotated spacially at an angle θ. The coordinate system x', y', z' is also rotates with angles

θ_{ij} as shown in the figure 4.2. Note that the angle θ_{ij} is defined so that the first index refers to the primed system and the second index to the unprimed system. Thus from the figure 4.2 we can write,

$$\begin{aligned}
\vec{r'} &= \vec{r} && (4.3)\\
i'x' + j'y' + k'z' &= ix + jy + kz \\
(i'x' + j'y' + k'z').i' &= (ix + jy + kz).i' \\
x' &= (i.i')\ x + (j.i')\ y + (k.i')\ z \\
x' &= cos\theta_{11}\ x + cos\theta_{12}\ y + cos\theta_{13}\ z \\
Similarly\ \ y' &= cos\theta_{21}\ x + cos\theta_{22}\ y + cos\theta_{23}\ z \\
z' &= cos\theta_{31}\ x + cos\theta_{32}\ y + cos\theta_{33}\ z && (4.4)
\end{aligned}$$

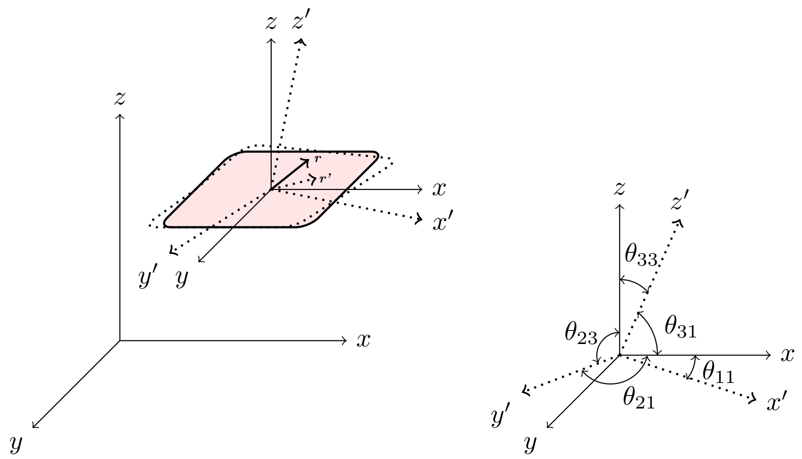

Figure 4.2: Rotation of a rigid body.

Equations 4.4 constitute a group of transformation equations from a set of coordinates x, y, z to a new set x', y', z' with direction cosines $cos\theta_{ij}$ as transformation coefficients. They form an example of a *linear or vector* transformation, defined by transformation equations of the form

$$\begin{aligned}
x' &= a_{11}\ x + a_{12}\ y + a_{13}\ z \\
y' &= a_{21}\ x + a_{22}\ y + a_{23}\ z \\
z' &= a_{31}\ x + a_{32}\ y + a_{33}\ z && (4.5)
\end{aligned}$$

where $a_{ij} = cos\theta_{ij}$. The equations 4.5 can be written in the form of matrix equation as

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.6}$$

$$[X'] \equiv [A]\,[X] \tag{4.7}$$

where $[X']$ is a column matrix with components (x', y', z'); $[X]$ is a similar column matrix with components (x, y, z) and $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the rotation matrix.

Consider a rigid body rotated in $x - y$ plane by an angle ϕ as shown in the figure 4.3. Then

$$\begin{aligned}
a_{11} &= cos\,\phi & a_{12} &= sin\,\phi & a_{13} &= 0 \\
a_{21} &= -sin\,\phi & a_{22} &= cos\,\phi & a_{23} &= 0 && (4.8)\\
a_{31} &= 0 & a_{32} &= 0 & a_{33} &= 1
\end{aligned}$$

The rotation matrix A can be written as

$$A = \begin{bmatrix} cos\,\phi & sin\,\phi & 0 \\ -sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \tag{4.9}$$

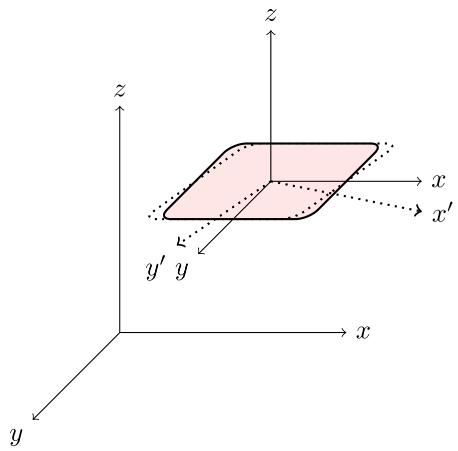

Figure 4.3: Rotation of a rigid body in $x - y$ plane.

$$\begin{aligned} A\tilde{A} &= \begin{bmatrix} cos\,\phi & sin\,\phi & 0 \\ -sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\,\phi & -sin\,\phi & 0 \\ sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ A\tilde{A} &= 1 \end{aligned} \tag{4.10}$$

From the equation 4.10, it is clear that $\tilde{A} = A^{-1}$. That is, the rotational matrix A is orthoganal and it can be taken as an operator. The matrix A operating on the components of a vector in the unprimed system yields the components of the vector in the primed system. Symbolically, the process can be written as,

$$\vec{r}\,' = A\vec{r} = \vec{r} \tag{4.11}$$

That is, the transformation matrix $[A]$ affects rotation of the rigid body with one point fixed has eigenvalue + 1. This is called Euler's theorem which states that *A general displacement of a rigid body with one point fixed is a rotation about some axis.*

A more general theorem than Euler's is proved by Chasles and it states that, *The most general displacement of rigid body is a translation of the rigid body plus rotation of the rigid body.*

It can be shown that, the orthogonal matrix whose determinant is -1 represents inversion and cannot represent physical displacement of a rigid body.

4.3 Euler angles

Eulerian angles are three rotations about three independent axes chosen in a certain successive way.

Consider a rigid body with initial system of axes xyz. Rotate the rigid body by by and angle ϕ counterclockwise about the z axis as shown in the figure 4.3, and the resultant coordinate system is labeled as $\xi\eta\zeta$. Then,

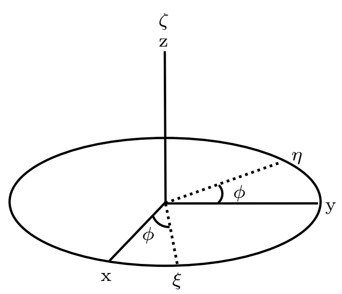

Figure 4.4: Rotation about z axis

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.12}$$

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = [A_\phi] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.13}$$

In the second stage, the intermediate axes, $\xi\eta\zeta$ are rotated about the ξ axis counterclockwise by an angle θ to

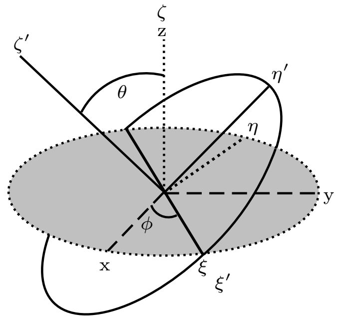

Figure 4.5: Rotation about ξ' axis

produce another intermediate set, $\xi'\eta'\zeta'$ axes as shown in the figure 4.3. The ξ' axis is at the intersection of the xy and $\xi'\eta'$ planes and is known as the *line of nodes*. The transformation then is written as,

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{4.14}$$

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = [A_\theta] \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{4.15}$$

Finally, $\xi'\eta'\zeta'$ axes are rotated counterclockwise by an angle ψ about the ζ' axis as shown in the figure 4.6 to produce the desired $x'y'z'$ system of axes.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} \tag{4.16}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [A_\psi] \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} \tag{4.17}$$

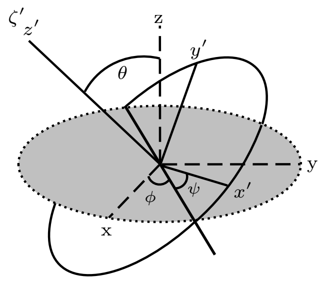

Figure 4.6: Rotation about ζ' axis

Then, the transformation of axes (xyz) to $(x'y'z')$ can be written by using the equations 4.13, 4.15 and 4.17 as,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [A_\phi]\,[A_\theta]\,[A_\psi] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.18}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [A] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.19}$$

where $[A] = [A_\phi]\,[A_\theta]\,[A_\psi]$. By using equations 4.12, 4.14 and 4.16,

$$A = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \tag{4.20}$$

It can be shown that $\tilde{A} = A^{-1}$, hence A is orthoganal.

4.4 Angular momentum and kinetic energy of motion about a point

Consider a rigid body moves with one point stationary, the total angular momentum about that point is

$$\vec{L} = \sum_i m_i(\vec{r_i} \times \vec{v_i}) \tag{4.21}$$

where $\vec{r_i}$ and $\vec{v_i}$ are the radius vector and velocity, respectively, of the ith particle relative to the given point. Since $\vec{r}$, is a fixed vector relative to the body, the velocity $\vec{v}$, with respect to the fixed frame arises solely from the rotational motion of the rigid body about the fixed point. Then,

$$\vec{v_i} = \vec{\omega} \times \vec{r_i}$$

where $\vec{\omega}$ is the angular velocity of the rigid body. The equation 4.21 becomes

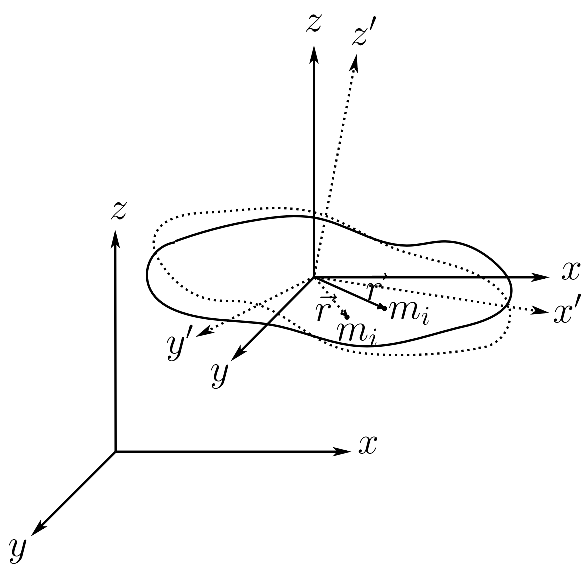

Figure 4.7: Rotation of a rigid body

$$L = \sum_i m_i[\vec{r_i} \times (\vec{\omega} \times \vec{r_i})] \tag{4.22}$$

[1] By using the vector triple product,

$$\begin{aligned}
\vec{L} &= \sum_i m_i\,[\vec{\omega}.(\vec{r_i}.\vec{r_i}) - \vec{r_i}.(\vec{r_i}.\vec{\omega})] \\
iL_x + jL_y + kL_z &= \sum_i m_i[(i\omega_x + j\omega_y + k\omega_z)(x_i^2 + y_i^2 + z_i^2) \\
&\quad -(ix_i + jy_i + kz_i)(x_i\omega_x + y_i\omega_y + z_i\omega_z)] \\
iL_x + jL_y + kL_z &= \sum_i m_i[(i\omega_x + j\omega_y + k\omega_z)(x_i^2 + y_i^2 + z_i^2) \\
&\quad -(ix_i^2\omega_x + ix_iy_i\omega_y + ix_iz_i\omega_z + jy_ix_i\omega_x + jy_i^2\omega_y \\
&\quad +jy_iz_i\omega_z + kz_ix_i\omega_x + kz_iy_i\omega_y + kz_i^2\omega_z)] \qquad (4.23)
\end{aligned}$$

By using 4.23, the components of angular momentum can be written as

$$\begin{aligned}
L_x &= \sum_i m_i[\omega_x(y_i^2 + z_i^2) - (x_iy_i\omega_y + x_iz_i\omega_z)] \\
L_y &= \sum_i m_i[\omega_y(x_i^2 + z_i^2) - (y_ix_i\omega_x + y_iz_i\omega_z)] \\
L_z &= \sum_i m_i[\omega_z(x_i^2 + y_i^2) - (z_ix_i\omega_x + z_iy_i\omega_y)]
\end{aligned}$$

$$\begin{aligned}
L_x &= \sum_i m_i(y_i^2 + z_i^2)\omega_x - \sum_i(m_ix_iy_i)\omega_y - \sum_i(m_ix_iz_i)\omega_z \\
L_y &= -\sum_i(m_iy_ix_i)\omega_x + \sum_i m_i(x_i^2 + z_i^2)\omega_y - \sum_i(m_iy_iz_i)\omega_z \qquad (4.24) \\
L_z &= -\sum_i(m_iz_ix_i)\omega_x - \sum_i(m_iz_iy_i)\omega_y + \sum_i m_i(x_i^2 + y_i^2)\omega_z
\end{aligned}$$

Equation 4.24 is rewritten as

$$\begin{aligned}
L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \\
\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} &= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad (4.25) \\
L &= \mathbf{I}\omega \qquad (4.26)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{I} &= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \\
&= \begin{bmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_ix_iy_i & -\sum_i m_ix_iz_i \\ -\sum_i m_iy_ix_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_iy_iz_i \\ -\sum_i m_iz_ix_i & -\sum_i m_iz_iy_i & \sum_i m_i(x_i^2 + y_i^2) \end{bmatrix} \qquad (4.27)
\end{aligned}$$

The matrix I acts as a linear operator to transforms $\vec{\omega}$ into $\vec{L}$. It has elements that behave as the elements of a second-rank tensor. Therefore **I** is usually called the moment of inertia tensor. The elements of moment of inertia tensor can be written as

$$I_{\alpha\beta} = \sum_i m_i(r_i^2\delta_{\alpha_i\beta_i} - \alpha_i\beta_i) \qquad \delta_{\alpha_i\beta_i} = 1 \;\; for \;\; \alpha_i = \beta_i \qquad (4.28)$$

$$\delta_{\alpha_i\beta_i} = 0 \;\; for \;\; \alpha_i \neq \beta_i$$

---

[1]

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}.(\vec{A}.\vec{C}) - \vec{A}.(\vec{B}.\vec{C})$$

If the body is regarded as continuous with mass density $\rho(\vec{r})$ and $\sum_i m_i = \int \rho(\vec{r})dV$ then, the tensor can be written as

$$I_{\alpha\beta} = \int \rho(\vec{r})(r_i^2\delta_{\alpha_i\beta_i} - \alpha_i\beta_i)dV \qquad \begin{array}{l}\delta_{\alpha_i\beta_i} = 1 \quad for \quad \alpha_i = \beta_i \\ \delta_{\alpha_i\beta_i} = 0 \quad for \quad \alpha_i \neq \beta_i\end{array} \tag{4.29}$$

[2] The kinetic energy of motion about a point is

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i v_i^2 \\ &= \frac{1}{2}\sum_i m_i \left[V_{CM} + (\vec{\omega}\times\vec{r_i})\right]^2 \\ &= \frac{1}{2}\sum_i m_i \left[V_{CM}^2 + 2V_{CM}(\vec{\omega}\times\vec{r_i}) + (\vec{\omega}\times\vec{r_i})^2\right] \\ T &= \frac{1}{2}\sum_i m_i V_{CM}^2 + V_{CM}\left(\vec{\omega}\times\sum_i m_i\vec{r_i}\right) + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r_i})^2 \end{aligned}$$

As distance $\vec{r_i}$ is measured from the CM, $\sum_i m_i\vec{r_i} = 0$ and $\sum_i m_i = M$. Then

$$\begin{aligned} T &= \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r_i})^2 \\ T &= T_t + T_r \end{aligned}$$

where $T_t = MV_{CM}^2/2$ is the kinetic energy of the rigid body due to translational motion and T_r is the kinetic energy of the rigid body due to rotational motion. Thus,

$$\begin{aligned} T_r &= \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r_i}).(\vec{\omega}\times\vec{r_i}) \\ &= \frac{1}{2}\sum_i m_i\vec{v_i}.(\vec{\omega}\times\vec{r_i}) \\ &= \frac{1}{2}\sum_i m_i\vec{\omega}.(\vec{r_i}\times\vec{v_i}) \\ T_r &= \frac{1}{2}\vec{\omega}.\sum_i(\vec{r_i}\times\vec{p_i}) \\ T_r &= \frac{1}{2}\vec{\omega}.\vec{L} = \frac{1}{2}\vec{\omega}.\mathbf{I}.\vec{\omega} = \frac{1}{2}\omega^2(\hat{n}.\mathbf{I}.\hat{n}) = \frac{1}{2}\omega^2 I \end{aligned} \tag{4.30}$$

where $\hat{n}$ is the unit vector in the ω direction and $I = [\tilde{n}]\mathbf{I}[n]$ is the moment of inertia of the body about the axis of rotation.

$$\begin{aligned} I &= \frac{2T_r}{\omega^2} \\ &= \frac{1}{\omega^2}\sum_i m_i(\vec{r}\times\vec{\omega}).(\vec{r}\times\vec{\omega}) \\ I &= \sum_i m_i(\vec{r}\times\hat{n}).(\vec{r}\times\hat{n}) \end{aligned} \tag{4.31}$$

4.5 The eigenvalues of the inertia tensor and the principal axis transformation

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \sum_i m_i(y_i^2+z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2+z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2+y_i^2) \end{bmatrix}$$

From the equation above equation, we can see that,

$$I_{xy} = I_{yx} \qquad I_{yz} = I_{zy} \qquad I_{xz} = I_{zx}$$

[2] $\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$

i.e., the inertia tensor I generally have nine components, only six of them will be independent - the three along the diagonal plus three of the off-diagonal elements. According to equation 4.28,

$$I_{\alpha\beta} = \sum_i m_i(r_i^2 \delta_{\alpha_i\beta_i} - \alpha_i\beta_i) \qquad \delta_{\alpha_i\beta_i} = 1 \;\; for \;\; \alpha_i = \beta_i$$

$$\delta_{\alpha_i\beta_i} = 0 \;\; for \;\; \alpha_i \neq \beta_i$$

$$\begin{aligned} I_{xx} = I_x &= \sum_i m_i(r_i^2 - x_i^2) = \sum_i m_i(y_i^2 + z_i^2) \\ I_{yy} = I_y &= \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + z_i^2) \\ I_{zz} = I_z &= \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + y_i^2) \end{aligned}$$

The inertia coefficients depend both upon the location of the origin of the body set of axes and upon the orientation of these axes with respect to the body. This symmetry suggests that there exists a set of coordinates in which the tensor is diagonal with the three principal values I_1, I_2 and I_3.
If R is the rotation matrix ($\vec{r}' = R\vec{r}$) , then the orthoganal transformation of the moment of inertial tensor I to I_D

$$\mathbf{I}_D = \tilde{R}\mathbf{I}R$$

This rotation can be expressed in terms of the Euler angles ϕ, θ, ψ as shown in equation 4.20. A proper choice of these angles will transform I into its diagonal form

$$\mathbf{I}_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{4.32}$$

i.e., like any real symmetric matrix, MI tensor I can be diagonalized by choosing appropriate symmetry axes. These are called principal axes of I. The diagonal matrix I_D is called principal moment of inertia. The eigen values I_1, I_2 and I_3 are called components of principal moment of inertia. The directions of x', y' and z' defined by the rotation matrix R are called the *principal axes*, or *eigen vectors* of the inertia tensor.
With principal moments of inertia, the components of L would involve only the corresponding component of ω, thus

$$L_1 = I_1\omega_1 \qquad L_2 = I_2\omega_2 \qquad L_3 = I_3\omega_3$$

The kinetic energy of the rigid body with origin of body axes at its CM can be written as,

$$T = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

We can understand the concept of principal axes through some geometrical considerations. The moment of inertia about a given axis has been defined in the equation 4.30 as

$$I = \hat{n}.\mathbf{I}.\hat{n}$$

where $\mathbf{I}$ is the inertia tensor and $\hat{n}$ is the unit vector along $\vec{\omega}$. Let the direction cosines of the axis be α, β and γ, then

$$\vec{n} = \alpha i + \beta j + \gamma k$$

$$\begin{aligned} I &= [\tilde{n}]\mathbf{I}[n] \\ &= \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \\ I &= I_{xx}\alpha^2 + I_{yy}\beta^2 + I_{zz}\gamma^2 + 2I_{xy}\alpha\beta + 2I_{yz}\beta\gamma + 2I_{zx}\alpha\gamma \end{aligned} \tag{4.33}$$

It is convenient to define a vector $\vec{\rho}$ by the equation

$$\begin{aligned} \vec{\rho} &= \frac{\vec{n}}{\sqrt{I}} \\ i\rho_1 + j\rho_2 + k\rho_3 &= i\frac{\alpha}{\sqrt{I}} + j\frac{\beta}{\sqrt{I}} + k\frac{\gamma}{\sqrt{I}} \end{aligned} \tag{4.34}$$

The magnitude of $\vec{\rho}$ is thus related to the moment of inertia about the axis whose direction is given by $\vec{n}$.

On substituting the values of α, β and γ from equation 4.34 to 4.33,

$$1 = I_{xx}\rho_1^2 + I_{yy}\rho_2^2 + I_{zz}\rho_3^2 + 2I_{xy}\rho_1\rho_2 + 2I_{yz}\rho_2\rho_3 + 2I_{zx}\rho_1\rho_3 \quad = \quad \vec{\rho}.\mathbf{I}.\vec{\rho} \tag{4.35}$$

Equation 4.35 is the equation of an ellipsoid designated as the *inertial ellipsoid.*
We can transform ρ_1, ρ_2, ρ_3 to a set of Cartesian axes With the principal axes of the ellipsoid along the new Cartesian axes, the equation of an ellipsoid takes its normal form as

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 \quad = \quad 1 \tag{4.36}$$

The principal moments of inertia I_1, I_2, I_3 determine the lengths of the axes of the inertia ellipsoid. If two of the roots of the secular equation are equal, the inertia ellipsoid thus has two equal axes and is an ellipsoid of revolution. If all three principal moments are equal, the inertia ellipsoid is a sphere.
If R_o is the radius of gyration,

$$I = MR_o^2$$

The vector $\vec{\rho}$ can be written as,

$$\vec{\rho} = \frac{\vec{n}}{R_o\sqrt{M}}$$

The radius vector to a point on the inertia ellipsoid is thus inversely proportional to the radius of gyration about the direction of the vector.

4.5.1 Classification of rigid bodies

1. If $I_1 \neq I_2 \neq I_3$, the rigid body is called asymmetric top.
2. If $I_1 = I_2 \neq I_3$, the body is called symmetric top.
3. If $I_1 = I_2 = I_3$, the body is called spherical top. Here one can choose any three mutually perpendicular axes as principal axes.

4.6 Theorems on moment of inertia

4.6.1 Theorem of perpendicular axes

The MI of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of inertias about any two perpendicular axes in the plane that intersects the first axis.

The components of moments of inertia tensor can be written as

$$I_{\alpha\beta} = \sum_i m_i(r_i^2\delta_{\alpha_i\beta_i} - \alpha_i\beta_i) \qquad \delta_{\alpha_i\beta_i} = 1 \;\; for \;\; \alpha_i = \beta_i$$

$$\delta_{\alpha_i\beta_i} = 0 \;\; for \;\; \alpha_i \neq \beta_i$$

If the lamina is rotating about z axis with its CM at origin,

$$I_{zz} = I_z \quad = \quad \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 = I_x + I_y$$

4.6.2 Theorem of parallel axis

The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its CM, together with the product of its mass and the square of the distance between the axes. From the figure 4.8, $r_i = R + r_i'$. MI of the rigid body about an axis passing through O is

$$\begin{aligned} I &= \sum_i m_i(r_i \times \hat{n})^2 \\ &= \sum_i m_i[(R \times \hat{n}) + (r_i' \times \hat{n})]^2 \\ &= \sum_i m_i(R \times \hat{n})^2 + 2(R \times \hat{n})\sum_i m_i(r_i' \times \hat{n}) + \sum_i m_i(r_i' \times \hat{n})^2 \end{aligned}$$

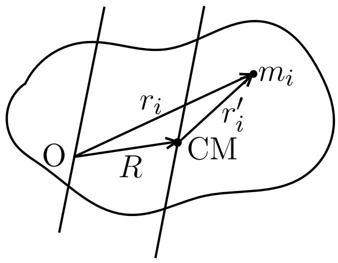

Figure 4.8: Rotation of a rigid body about an axis passing through O

About the center of mass $\sum_i m_i r'_i = 0$,

$$I = \sum_i m_i (R \times \hat{n})^2 + \sum_i m_i (r'_i \times \hat{n})^2$$
$$I = MR^2 sin^2\theta + I_{CM}$$

where θ is the angle between $\vec{\omega}$ direction and $\vec{R}$ direction. That is the moment of inertia about a given axis is equal to the moment of inertia about a parallel axis through the center of mass plus the moment of inertia of the body, as if concentrated at the center of mass, with respect to the original axis.

4.7 The Euler equations of motion

For the rotational motion about a fixed point or the center of mass, the direct Newtonian approach leads to a set of equations known as Euler's equations of motion. For a rigid body, the general motion is both traslational and rotational. Therefore a frame attached to a rigid body is a non inertial frame. With reference to the fixed frame in space, the general operator equation can be written as

$$\left[\frac{d(\ )}{dt}\right]_{fixed} = \left[\frac{d(\ )}{dt}\right]_{body} + \vec{\omega} \times (\ )$$

where, ( ) contain any vector operator. Thus,

$$\left[\frac{d\vec{L}}{dt}\right]_{fixed} = \left[\frac{d\vec{L}}{dt}\right]_{body} + \vec{\omega} \times \vec{L}$$

$$N^{(e)} = \left[\frac{d\vec{L}}{dt}\right]_{body} + \vec{\omega} \times \vec{L} \tag{4.37}$$

The componets of equation 4.37 along x, y and z directions are

$$\begin{aligned} N_x &= \frac{d\vec{L_x}}{dt} + (\omega_y L_z - \omega_z L_y) \\ N_y &= \frac{d\vec{L_y}}{dt} + (\omega_z L_x - \omega_x L_z) \\ N_z &= \frac{d\vec{L_z}}{dt} + (\omega_x L_y - \omega_y L_x) \end{aligned} \tag{4.38}$$

If the body axes are taken as the principal axes, relative to reference point and if, I_1, I_2, I_3 are the principal moment of inertia along x, y, z directions, then

$$L_x = I_1 \omega_x \qquad L_y = I_2 \omega_y \qquad L_z = I_3 \omega_z$$

The set of equations 4.38 becomes,

$$\begin{aligned} N_x &= I_1 \dot{\omega}_x - (I_2 - I_3)\omega_y \omega_z \\ N_y &= I_2 \dot{\omega}_y - (I_3 - I_1)\omega_x \omega_z \\ N_z &= I_3 \dot{\omega}_z - (I_1 - I_2)\omega_x \omega_y \end{aligned} \tag{4.39}$$

Equations 4.35 are Euler's equations of motion for a rigid body with one point fixed.

4.8 Torque-free motion of a rigid body

Euler's equations of motion for a rigid body with one point fixed are

$$\begin{aligned} N_x &= I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z \\ N_y &= I_2\dot{\omega}_y - (I_3 - I_1)\omega_x\omega_z \\ N_z &= I_3\dot{\omega}_z - (I_1 - I_2)\omega_x\omega_y \end{aligned} \tag{4.40}$$

In the absence of any net torques, ($N = 0$) they reduce to

$$\begin{aligned} I_1\dot{\omega}_x &= (I_2 - I_3)\omega_y\omega_z \\ I_2\dot{\omega}_y &= (I_3 - I_1)\omega_x\omega_z \\ I_3\dot{\omega}_z &= (I_1 - I_2)\omega_x\omega_y \end{aligned} \tag{4.41}$$

4.8.1 Poinsot's geometrical construction

Consider a coordinate system oriented along the principal axes of the body, but whose axes measure the components of a vector $\vec{\rho}$ along the instantaneous axis of rotation as

$$\vec{\rho} = \frac{\vec{n}}{\sqrt{I}} = \frac{\vec{\omega}}{\omega\sqrt{I}} = \frac{\vec{\omega}}{\sqrt{2T}} \tag{4.42}$$

In ρ space, we define a function $F(\rho)$ called surfaces of constant as

$$F(\rho) = \vec{\rho}.\mathbf{I}.\vec{\rho} \tag{4.43}$$

The surfaces of constatnt $F(\rho)$ are ellipsoid and $F = 1$ being the inertia ellipsoid. As the direction of the axis of rotation changes in time, the parallel vector $\vec{\rho}$ moves accordingly, its tip always defining a point on the inertia ellipsoid.

$$\nabla F(\rho) = \nabla(\vec{\rho}.\mathbf{I}.\vec{\rho}) \tag{4.44}$$

By using equation 4.35, equation 4.44 can be written as

$$\begin{aligned} \nabla F(\rho) &= \left(i\frac{\delta}{\delta\rho_1} + j\frac{\delta}{\delta\rho_2} + k\frac{\delta}{\delta\rho_3}\right) \\ &\quad (I_{xx}\rho_1^2 + I_{yy}\rho_2^2 + I_{zz}\rho_3^2 + 2I_{xy}\rho_1\rho_2 + 2I_{yz}\rho_2\rho_3 + 2I_{zx}\rho_1\rho_3) \\ &= 2i(I_{xx}\rho_1 + I_{xy}\rho_2 + I_{xz}\rho_3) + 2j(I_{yy}\rho_2 + I_{xy}\rho_1 + I_{yz}\rho_3) \\ &\quad +2k(I_{zz}\rho_3 + I_{yz}\rho_2 + I_{xz}\rho_1) \\ &= 2\begin{bmatrix} \rho_1 & \rho_2 & \rho_3 \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \\ \nabla F(\rho) &= 2\vec{\rho}.\mathbf{I} = 2\mathbf{I}.\vec{\rho} \end{aligned}$$

On substituting for $\vec{\rho}$ by using equation 4.42,

$$\nabla F(\rho) = = \frac{2\mathbf{I}.\vec{\omega}}{\sqrt{2T}} = \sqrt{\frac{2}{T}}\vec{L}$$

Thus, the $\vec{\omega}$ will always move such that the corresponding normal to the inertia ellipsoid is in the direction of the angular momentum $\vec{L}$. The tangent plane to the ellipsoid at the tip of $\vec{\rho}$ is called the *invariable plane* (Figure 4.9). The distance between the origin of the inertia ellipsoid and the inavrible plane is

$$\rho\, cos\,(\vec{\rho}.\vec{L}) = \frac{\vec{\rho}.\vec{L}}{L}$$

By using equation 4.42,

$$\frac{\vec{\rho}.\vec{L}}{L} = \frac{\vec{\omega}.\vec{L}}{L\sqrt{2T}} = \frac{\sqrt{2T}}{L}$$

Since both T and L are constants, the distance from the origin of the ellipsoid to the invariable plane remains constant and the point of contact is defined by the position of ρ. Thus the inertia ellipsoid rolls without slipping on the invariable plane. The curve traced out by the point of contact (tip of $\vec{\rho}$) on the inertia ellipsoid is known as the *polhode*, while the cnrve traced out by the point of contact (tip of $\vec{\rho}$) on the invariable plane is called the *herpolhode*.

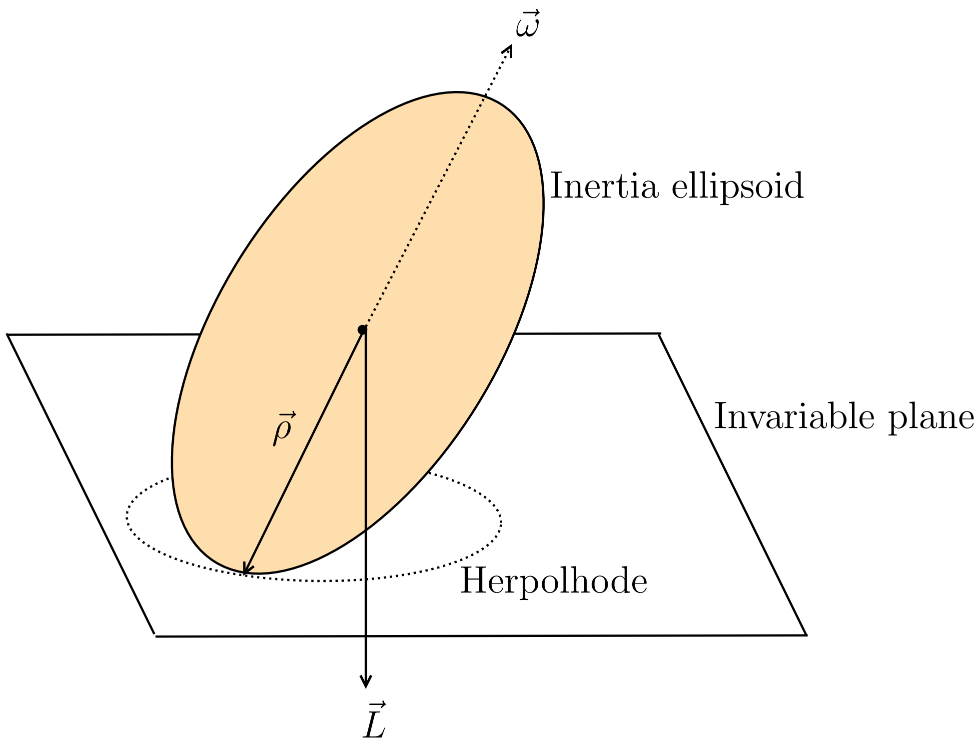

Figure 4.9: The motion of the inertia ellipsoid relative to the invariable plane.

4.8.2 Rotation of a symmetrical body

In case of a symmetrical body, the inertia ellipsoid is an ellipsoid of revolution, so that the polhode on the ellipsoid is clearly a circle about the symmetry axis. The herpolhode on the invariable plane is also a circle. An observer fixed in the body sees the angular velocity vector $\vec{\omega}$ move on the surface of a cone called the *body cone* whose intersection with the inertia ellipsoid is the polhode. An observer fixed in the space axes sees the $\vec{\omega}$ move on the surface of a space cone whose intersection with the invariable plane is the herpolhode. Thus, the free motion of the symmetrical rigid body is rolling of the body cone on the space cone.
If the moment of inertia about the symmetry axis is less than that about the other two principal axes, the inertia ellipsoid is prolate and the body cone rolls outside the space cone. When the moment of inertia about the symmetry axis is the greater, the ellipsoid is oblate and the body cone rolls inside the space cone.
In the absence of net torques, Euler's equations can be written as

$$\begin{aligned} I_1\dot{\omega}_x &= (I_2 - I_3)\omega_y\omega_z \\ I_2\dot{\omega}_y &= (I_3 - I_1)\omega_x\omega_z \\ I_3\dot{\omega}_z &= (I_1 - I_2)\omega_x\omega_y \end{aligned}$$

Let the symmetry axis of the body be taken as the L_z principal axis, so that $I_1 = I_2$. Euler's equations reduce then to

$$I_1\dot{\omega}_x = (I_1 - I_3)\omega_y\omega_z \tag{4.45}$$

$$I_1\dot{\omega}_y = (I_3 - I_1)\omega_x\omega_z \tag{4.46}$$

$$I_3\dot{\omega}_z = 0 \implies \omega_z = constant$$

Equations 4.45 and 4.46 can be written as

$$\dot{\omega}_x = \frac{(I_1 - I_3)\omega_z}{I_1}\omega_y = \Omega\omega_y \tag{4.47}$$

$$\dot{\omega}_y = -\frac{(I_1 - I_3)\omega_z}{I_1}\omega_x = -\Omega\omega_x \tag{4.48}$$

where $\Omega = \frac{(I_1-I_3)\omega_z}{I_1}$. On differentiating equation 4.47 and substituting for $\dot{\omega}_y$ by using equation 4.48,

$$\ddot{\omega}_x = \Omega\dot{\omega}_y = -\Omega^2\omega_x$$

$$\ddot{\omega}_x + \Omega^2\omega_x = 0 \tag{4.49}$$

Equation 4.49 is the standard diffeiential equation for simple harmonic motion. The solution of the equation can be written as

$$\omega_x = A\,cos\,\Omega t \tag{4.50}$$
$$\dot{\omega}_x = -\Omega\,A\,sin\,\Omega t \tag{4.51}$$

Equation 4.51 in equation 4.47,

$$\omega_x = A\,sin\,\Omega t \tag{4.52}$$

Equations 4.50 and 4.52 shows that the vector $i\omega_x + j\omega_y$ has a constant magnitude and rotates uniformly about the z axis of the body with the angular frequency Ω.

$$|\omega| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \sqrt{A^2 + \omega_z^2} \tag{4.53}$$

Hence, the total angular velocity $\vec{\omega}$ is also constant in magnitude and precesses about the z axis with the same frequency, exactly as predicted by the Poinsot construction.

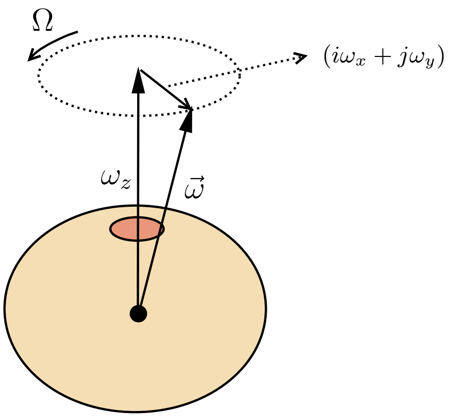

Figure 4.10: Precession of the angular velocity about the axis of symmetry in the force-free motion of a symmetrical rigid body.

Reference books:

- Classical Mechanics - P V Panat
- Classical Mechanics, 3rd Ed - H.Goldstein, C.Poole and J.Safko
- Introduction to classical mechanics - R K Takwale and P S Puranik
- Classical Mechanics - N C Rana and P S Joag

Chapter 5

Small oscillations of a mechanical system

The theory of small oscillations finds widespread physical applications in acoustics, molecular spectra, vibrations of mechanisms, and coupled electrical circuits. If the deviations of the system from stable equilibrium conditions are small enough, the motion can generally be described as that of a system of coupled linear harmonic oscillators. An equilibrium position is classified as stable if a small disturbance of the system from equilibrium results only in small bounded motion about the rest position. The equilibrium is unstable if an infinitesimal disturbance eventually produces unbounded motion. If V is a minimum at equilibrium, any deviation from this position will produce an increase in V. By the conservation of energy, the velocities must then decrease and eventually come to zero, indicating bound motion. If V decreases as the result of some departure from equilibrium, the kinetic energy and the velocities increase indefinitely, corresponding to unstable motion.

5.1 Study of small oscillations using generalized coordinates

Consider scleronomic and holonomic conservative dynamic system having generalized coordinates $q_1, q_2, q_3, ..., q_n$. In the stable state kinetic energy of the system $T = 0$. Then the Lagrangian L of the system is

$$L = T - V = -V \tag{5.1}$$

The Lagrangian equation is

$$-\frac{d}{dt}\left(\frac{\delta V}{\delta \dot{q}_i}\right) + \frac{\delta V}{\delta q_i} = 0, \qquad i = 1, 2, 3, ..., n. \tag{5.2}$$

If the configuration is initially at the equilibium position, with initial velocities $\dot{q}_i = 0$, then the system will continue in equilibrium indefinitely. The generalized coordinates at the equilibrium is written as $q_{10}, q_{20}, q_{30}, ..., q_{n0}$ and the equation 5.2 becomes

$$Q_i = -\left(\frac{\delta V}{\delta q_i}\right)_0 = 0 \tag{5.3}$$

Thus the potential energy has an extremum value at the equilibrium configuration of the system. For bound motion V is minimum at equilibrium. Let us disturb the system by giving small displacements η such that, $q_i = q_{oi} + \eta_i$, for the slightest possible perturbation, we can write

$$V(q_1, q_2, q_3, ..., q_n) > V(q_{01}, q_{02}, q_{03}, ..., q_{0n}) \tag{5.4}$$

By setting

$$V(q_{01}, q_{02}, q_{03}, ..., q_{0n}) = 0 \tag{5.5}$$

without losing any generality, the inequality 5.4 can be written as

$$V(q_1, q_2, q_3, ..., q_n) > 0$$

Then we expand a potential function using Taylor series around its minimum. We then obtain,

$$V(q_1, q_2, q_3, ..., q_n) = V(q_{01}, q_{02}, q_{03}, ..., q_{0n}) + \sum_i \left(\frac{\delta V}{\delta q_i}\right)_0 \eta_i + \frac{1}{2}\sum_{ij}\left(\frac{\delta^2 V}{\delta q_i \delta q_j}\right)_0 \eta_i \eta_j + \frac{1}{6}\sum_{ijk}\left(\frac{\delta^2 V}{\delta q_i \delta q_j \delta q_k}\right)_0 \eta_i \eta_j \eta_k + ... \tag{5.6}$$

By using equation 5.3 and 5.5 and by neglectiong the higher order terms, the equation 5.6 becomes

$$V(q_1, q_2, q_3, ..., q_n) = \frac{1}{2}\sum_{ij}\left(\frac{\delta^2 V}{\delta q_i \delta q_j}\right)_0 \eta_i\eta_j$$
$$V = \frac{1}{2}\sum_{ij} V_{ij}\eta_i\eta_j \tag{5.7}$$

where

$$V_{ij} = \left(\frac{\delta^2 V}{\delta q_i \delta q_j}\right)_0$$

The equation 5.7 can be written in the matrix form as

$$V = \frac{1}{2}\left[\begin{array}{ccc} \eta_1 & \eta_2 & ....\eta_n \end{array}\right]\left[\begin{array}{ccc} V_{11} & V_{12} & ....V_{1n} \\ V_{21} & V_{22} & ....V_{2n} \\ V_{31} & V_{32} & ....V_{3n} \\ . & & \\ . & & \\ . & & \\ V_{n1} & V_{n2} & ....V_{nn} \end{array}\right]\left[\begin{array}{c} \eta_1 \\ \eta_2 \\ \eta_3 \\ . \\ . \\ . \\ \eta_n \end{array}\right]$$
$$V = \frac{1}{2}\tilde{\eta}\,\mathbf{V}\eta \tag{5.8}$$

where $\mathbf{V}$ is a tensor of $(n-1)$ rank. When the constraint equations are scleronomic, the kinetic energy of the system is given by

$$T = \frac{1}{2}\sum_i\sum_j m_{ij}\dot{q}_i\dot{q}_j \qquad \text{refer equations 3.44 and 3.46} \tag{5.9}$$

where

$$m_{ij} = \sum_p m_p \frac{\delta\vec{r}_p}{\delta q_i}\frac{\delta\vec{r}_p}{\delta q_j} \tag{5.10}$$

In the equation 5.10, p is summing over to number of particles. Since,

$$q_i = q_{oi} + \eta_i$$
$$\dot{q}_i = \dot{\eta}_i$$

The equation 5.9 becomes,

$$T = \frac{1}{2}\sum_{ij} m_{ij}\dot{\eta}_i\dot{\eta}_j \tag{5.11}$$

From equation 5.10, we see that, $m_{ij} = m_{ij}(q_1, q_2, .., q_n)$. We expand m_{ij} using Taylor series around the equilibrium position as,

$$m_{ij}(q_1, q_2, q_3, ..., q_n) = m_{ij}(q_{01}, q_{02}, q_{03}, ..., q_{0m}) + \sum_l \left(\frac{\delta m_{ij}}{\delta q_k}\right)_0 \eta_k + ... \tag{5.12}$$

As equation 5.11 is already quadratic in the $\dot{\eta}_i$, the lowest nonvanishing approximation to T is obtained by dropping all terms in equation 5.11 except first term. Therefore,

$$m_{ij}(q_1, q_2, q_3, ..., q_n) = m_{ij}(q_{01}, q_{02}, q_{03}, ..., q_{0n})$$

The equation 5.11 becomes,

$$T = \frac{1}{2}\sum_{ij}(m_{ij})_0\dot{\eta}_i\dot{\eta}_j$$

Denoting the constant values of the $(m_{ij})_0 = T_{ij}$, we can therefore write the kinetic energy as

$$T = \frac{1}{2}\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j \tag{5.13}$$

The equation 5.7 can be written in the matrix form as

$$T = \frac{1}{2}\begin{bmatrix} \dot{\eta_1} & \dot{\eta_2} & \dot{\eta_3} & \ldots\ldots\ldots\ldots\dot{\eta_n} \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} & \ldots\ldots\ldots\ldots\ldots T_{1n} \\ T_{21} & T_{22} & \ldots\ldots\ldots\ldots\ldots T_{2n} \\ T_{31} & T_{32} & \ldots\ldots\ldots\ldots\ldots T_{3n} \\ . & & \\ . & & \\ . & & \\ T_{n1} & T_{n2} & \ldots\ldots\ldots\ldots\ldots T_{nn} \end{bmatrix}\begin{bmatrix} \dot{\eta_1} \\ \dot{\eta_2} \\ \dot{\eta_3} \\ . \\ . \\ . \\ \dot{\eta_n} \end{bmatrix}$$

$$T = \frac{1}{2}\tilde{\dot{\eta}}\,\mathbf{T}\dot{\eta} \tag{5.14}$$

where $\mathbf{T}$ is a tensor of $(n-1)$ rank. By using equations 5.7 and 5.13, the Lagrangian can be written as

$$L = \frac{1}{2}\left(\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j - \sum_{ij} V_{ij}\eta_i\eta_j\right)$$

Taking η's as the general coordinates, the Lagrangian can be written as

$$L = \frac{1}{2}\left(\sum_{ij} T_{ij}\dot{\eta}_j^2 - \sum_{ij} V_{ij}\eta_j^2\right) \tag{5.15}$$

$$\frac{\delta L}{\delta\dot{\eta}_j} = \sum_{ij} T_{ij}\dot{\eta}_j$$

$$\frac{\delta L}{\delta\eta_j} = -\sum_{ij} V_{ij}\eta_j$$

By treating η_i as generalized coordinates and $\dot{\eta}_i$ as generalized velocities, the Lagrange's equations can be written as

$$\frac{d}{dt}\sum_{ij} T_{ij}\dot{\eta}_j + \sum_{ij} V_{ij}\eta_j = 0$$

$$\sum_{ij}(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j) = 0 \tag{5.16}$$

By using equations 5.8 and 5.14, the Lagrangian can also be written as

$$L = \frac{1}{2}\tilde{\dot{\eta}}\,\mathbf{T}\dot{\eta} - \frac{1}{2}\tilde{\eta}\,\mathbf{V}\eta \tag{5.17}$$

5.2 Normal coordinates and normal modes

Consider scleronomic and holonomic conservative dynamic system having generalized coordinates η_i. If the system is displaced slightly from equilibrium and then released, the system performs small oscillations about the equilibrium, the Lagrangian equations for the system are

$$\sum_{ij}(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j) = 0 \tag{5.18}$$

The above equations of motion are linear differential equations with constant coefficients. The solution of the equations can be in the form

$$\eta_j = Ca_j e^{-i\omega t} \tag{5.19}$$

On substituting for $\ddot{\eta}_j$ and η_j by using equation 5.19 to the equation 5.18,

$$\sum_{ij}\left[T_{ij}(-\omega^2 Ca_j e^{-i\omega t}) + V_{ij}Ca_j e^{-i\omega t}\right] = 0$$

$$\sum_{ij}\left[V_{ij} - \omega^2 T_{ij}\right]a_j = 0 \tag{5.20}$$

The equation 5.20 can be written in the matrix form as

$$\begin{bmatrix} V_{11}-\omega^2T_{11} & V_{12}-\omega^2T_{12} & ....V_{1n}-\omega^2T_{1n} \\ V_{21}-\omega^2T_{21} & V_{22}-\omega^2T_{22} & ....V_{2n}-\omega^2T_{2n} \\ V_{31}-\omega^2T_{31} & V_{32}-\omega^2T_{32} & ....V_{3n}-\omega^2T_{3n} \\ . & & \\ . & & \\ . & & \\ V_{m1}-\omega^2T_{n1} & V_{n2}-\omega^2T_{n2} & ....V_{nn}-\omega^2T_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_n \end{bmatrix} = 0$$

These n equations have a nontrivial solution if

$$\begin{bmatrix} V_{11}-\omega^2T_{11} & V_{12}-\omega^2T_{12} & ....V_{1m}-\omega^2T_{1n} \\ V_{21}-\omega^2T_{21} & V_{22}-\omega^2T_{22} & ....V_{2n}-\omega^2T_{2n} \\ V_{31}-\omega^2T_{31} & V_{32}-\omega^2T_{32} & ....V_{3n}-\omega^2T_{3n} \\ . & & \\ . & & \\ . & & \\ V_{n1}-\omega^2T_{n1} & V_{n2}-\omega^2T_{n2} & ....V_{nn}-\omega^2T_{nn} \end{bmatrix} = 0$$

$$\sum_{ij} V_{ij}-\omega^2T_{ij} = 0 \tag{5.21}$$

The equation 5.21 gives polynomial in ω^2 of order n and has n roots. These roots are real and positive. These n roots may be all distinct or some of them may be same. If k root are identical, then we say that the system is k - fold degeneracy.

Thus the solution of the form given in the equation 5.19 not for one frequency, but in general for a set of n frequencies. A complete solution of the equations of motion therefore involves a superposition of oscillations with all the allowed frequencies. Thus, if the system is displaced slightly from equilibrium and then released, the system performs small oscillations about the equilibrium with the frequencies $\omega_1, \omega_2, \omega_3, ..., \omega_n$. The solutions of the equation are therefore often designated as the *frequencies of free vibration* or as the *resonant frequencies* of the system.

The general solution of the equations of motion may be written as

$$\eta_i = C_k a_{jk} e^{-i\omega_k t} \tag{5.22}$$

where C_k is the complex scale factor for each frequency.

The solution for each generalized coordinate given in equation 5.22, is summing over all of the frequencies ω_k, satisfying the secular equation 5.21. Unless the frequencies are rational fractions of each other, η_i never repeats its initial value and is therefore not itself a periodic function of time. However, it is possible to transform from the η_i to a new set of generalized coordinates that are all simple periodic functions of time. These set of generalized coordinates are known as the *normal coordinates.* We define a new set of coordinates ξ_j as,

$$\eta_i = \sum_j a_{ij}\xi_j \tag{5.23}$$

Interms of matrix equation,

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ . \\ . \\ . \\ \eta_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ....a_{1n} \\ a_{21} & a_{22} & ....a_{2n} \\ a_{31} & a_{32} & ....a_{3n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & a_{n2} & ....a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ . \\ . \\ . \\ \xi_n \end{bmatrix}$$

$$\eta = \mathbf{A}\,\xi \quad \text{and} \quad \dot{\eta} = \mathbf{A}\,\dot{\xi} \tag{5.24}$$

where $\mathbf{A}$ is a tensor of $(n-1)$ rank. The potential energy is written in the matrix notation as

$$V = \frac{1}{2}\tilde{\eta}\mathbf{V}\eta \quad \text{refer equation 5.8}$$

By using equation 5.24,

$$V = \frac{1}{2}\widetilde{\mathbf{A}\xi}\mathbf{V}\mathbf{A}\xi$$

$$V = \frac{1}{2}\tilde{\xi}\tilde{\mathbf{A}}\mathbf{V}\mathbf{A}\xi \tag{5.25}$$

Since

$$\tilde{\mathbf{A}}\mathbf{V}\mathbf{A} = \begin{bmatrix} V_{11} & 0 & 0 & ....0 \\ 0 & V_{22} & 0 & ....0 \\ 0 & 0 & V_{33} & ....0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & ....V_{nn} \end{bmatrix} = [\lambda] \tag{5.26}$$

Then the equation 5.25 reduces to

$$V = \frac{1}{2}\tilde{\xi}\lambda\xi \tag{5.27}$$

The kinetic energy is written in the matrix notation as

$$T = \frac{1}{2}\tilde{\dot{\eta}}\mathbf{T}\dot{\eta} \quad \text{refer equation 5.14}$$

By using equation 5.24,

$$T = \frac{1}{2}\widetilde{\mathbf{A}\dot{\xi}}\mathbf{T}\mathbf{A}\dot{\xi}$$

$$T = \frac{1}{2}\tilde{\dot{\xi}}\tilde{\mathbf{A}}\mathbf{T}\mathbf{A}\dot{\xi} \tag{5.28}$$

It was shown that

$$\tilde{\mathbf{A}}\mathbf{T}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & ....0 \\ 0 & 1 & 0 & ....0 \\ 0 & 0 & 1 & ....0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & ....1 \end{bmatrix} = [\mathbf{1}] \tag{5.29}$$

Then the equation 5.25 reduces to

$$T = \frac{1}{2}\tilde{\dot{\xi}}\dot{\xi}$$

$$= \frac{1}{2}\begin{bmatrix} \dot{\xi}_1 & \dot{\xi}_2 & \dot{\xi}_3 & ....\dot{\xi}_n \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ . \\ . \\ . \\ \dot{\xi}_n \end{bmatrix}$$

$$T = \frac{1}{2}\left[\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2 + ... + \dot{\xi}_n^2\right] = \frac{1}{2}\sum \dot{\xi}_k^2 \tag{5.30}$$

When the matrix $\mathbf{T} = [1]$, it can be shown that

$$[\lambda] = \begin{bmatrix} \omega_1^2 & 0 & 0 & ....0 \\ 0 & \omega_2^2 & 0 & ....0 \\ 0 & 0 & \omega_3^2 & ....0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & ....\omega_n^2 \end{bmatrix}$$

Then equation 5.27 can be written as

$$V = \frac{1}{2}\begin{bmatrix} \xi_1 & \xi_2 & \xi_3 & ....\xi_n \end{bmatrix} \begin{bmatrix} \omega_1^2 & 0 & 0 & ....0 \\ 0 & \omega_2^2 & 0 & ....0 \\ 0 & 0 & \omega_3^2 & ....0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & ....\omega_n^2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ . \\ . \\ . \\ \xi_n \end{bmatrix}$$

$$V = \frac{1}{2}\left[\omega_1^2\xi_1^2 + \omega_2^2\xi_2^2 + \omega_3^2\xi_3^2 + ... + \omega_n^2\xi_n^2\right] = \frac{1}{2}\sum \omega_k^2\xi_k^2 \tag{5.31}$$

By using equations 5.30 and 5.31, we can write the Lagrangian as

$$\begin{aligned} L &= \frac{1}{2}(\dot{\xi}_k^2 - \omega_k^2 \xi_k^2) \\ \frac{\delta L}{\delta \dot{\xi}_k} &= \dot{\xi}_k \qquad \text{and} \qquad \frac{\delta \mathrm{L}}{\delta \mathrm{\xi_k}} = -\mathrm{\omega_k^2 \xi_k} \end{aligned}$$

Then the Lagrangian equation can be written as

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\xi}_k}\right) - \frac{\delta L}{\delta \xi_k} = \ddot{\xi}_k + \omega_k^2 \xi_k = 0 \tag{5.32}$$

The solution for the equation 5.32 can be written as

$$\xi_k = C_k e^{-i\omega_k t} \tag{5.33}$$

Thus from the equation 5.33, we can see that each normal cordinate ξ_k is simply periodic function involving only one of the resonant frequency. The modes of vibration corresponding to eaeh normal coordinates ξ_k are called *normal modes of vibration.* All of the particles in each mode vibrate with the same frequency and with the same phase, and the relative amplitudes being determined by the matrix elements a_{jk}. The complete motion is then built up out of the sum of the normal modes weighted with appropriate amplitude and phase factors contained in the $C_k' s$

5.3 Free vibrations of CO_2 molecule

CO_2 molecule is a linear symmetrical triatomic molecule. In the equilibrium configuration of the molecule, two oxygen atoms are symmetrically located on each side of carbon atom as shown in the figure 5.1. All three atoms are on one straight line, the equilibrium distances apart being denoted by b. For simplicity, we shall first consider only vibrations along the line of the molecule ($y_i = 0, z_i = 0$), and the actual complicated interatomic potential will be approximated by two springs of force constant k joining the three atoms. There are three coordinates (x_1, x_2, x_3) marking the position of the three atoms on the line. In these coordinates, the potential energy is

$$V = \frac{k}{2}[(x_2 - x_1) - (x_{02} - x_{01})]^2 + \frac{k}{2}[(x_3 - x_2) - (x_{03} - x_{02})]^2$$

$$V = \frac{k}{2}[(x_2 - x_{02}) - (x_1 - x_{01})]^2 + \frac{k}{2}[(x_3 - x_{03}) - (x_2 - x_{02})]^2 \tag{5.34}$$

The η coordinates can be written as

$$\eta = x_i - x_{0i} \tag{5.35}$$

Equation 5.35 in equation 5.34,

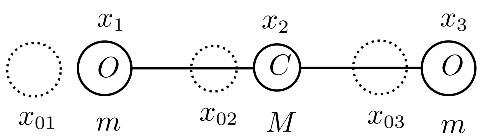

Figure 5.1: Model for a CO_2 molecule.

$$\begin{aligned} V &= \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2 \\ &= \frac{k}{2}(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3) \\ &= \frac{1}{2}\begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \end{bmatrix} \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \end{aligned} \tag{5.36}$$

Hence the $\mathbf{V}$ tensor has the form

$$\mathbf{V} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \tag{5.37}$$

The total kinetic energy of the molecule is

$$T = \frac{m}{2}\dot{\eta}_1^2 + \frac{M}{2}\dot{\eta}_2^2 + \frac{m}{2}\dot{\eta}_3^2 \tag{5.38}$$

where m and M are the masses of oxyzen and carbon molecules respectively. The equation 5.34 can be written in the matrix form as,

$$T = \frac{1}{2}\begin{bmatrix} \dot{\eta}_1 & \dot{\eta}_2 & \dot{\eta}_3 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{bmatrix}$$

Then the **T** tensor becomes

$$\mathbf{T} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \tag{5.39}$$

By using equations 5.37 and 5.39, the secuar equation can be written as

$$\begin{aligned} \left[\mathbf{V} - \omega^2\mathbf{T}\right] &= 0 \\ \begin{bmatrix} k-\omega^2 m & -k & 0 \\ -k & 2k-\omega^2 M & -k \\ 0 & -k & k-\omega^2 m \end{bmatrix} &= 0 \qquad (5.40) \\ (k-\omega^2 m)\left[(2k-\omega^2 M)(k-\omega^2 m) - k^2\right] - k^2(k-\omega^2 m) &= 0 \\ (k-\omega^2 m)\left[(2k-\omega^2 M)(k-\omega^2 m) - 2k^2\right] &= 0 \\ \omega^2(k-\omega^2 m)\left[k(M+2m) - \omega^2 mM\right] &= 0 \qquad (5.41) \end{aligned}$$

The solutions of the equation 5.41 are

$$\begin{aligned} \omega^2 &= 0 \quad \Longrightarrow \quad \omega_1 = 0 \\ k - \omega^2 m &= 0 \quad \Longrightarrow \quad \omega_2 = \sqrt{\frac{k}{m}} \\ k(M+2m) - \omega^2 mM &= 0 \quad \Longrightarrow \quad \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)} \end{aligned}$$

The first eigen value $\omega_1 = 0$ indicates that the molecule may be translated rigidly along its axis without any change in the potential energy. Since the restoring force against such motion is zero, the effective frequency must also vanish. We have made the assumption that the molecule has three degrees of freedom for vibrational motion, whereas in reality one of them is a rigid body degree of freedom.
The resonant frequency, ω_2 will be recognized as the well-known frequency of oscillation for a mass m suspended by a spring of force constant k. We are therefore led to expect that only the end atoms are vibrate and the center atom remains stationary in this mode. It is only in the third mode of vibration, ω_3, that the mass M

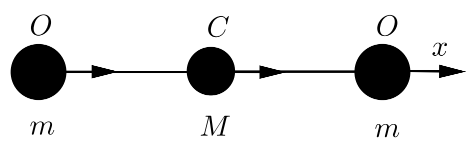

Figure 5.2: Mode1: Translational motion of CO_2 molecule.

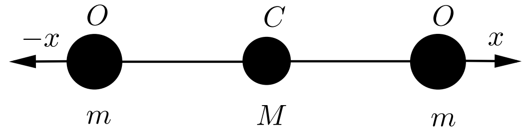

Figure 5.3: Mode2: Symmetric stretching of CO_2 molecule.

can participate in the oscillatory motion. These predictions are verified by examining the eigenvectors for the three normal modes. The eigen vectors are given by

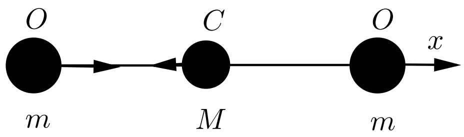

Figure 5.4: Mode3: Asymmetric stretching of CO_2 molecule.

$$\begin{aligned}
\left[\mathbf{V} - \omega_j^2 \mathbf{T}\right] a_{ij} &= 0 \\
\begin{bmatrix} k-\omega_j^2 m & -k & 0 \\ -k & 2k-\omega_j^2 M & -k \\ 0 & -k & k-\omega_j^2 m \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} &= 0 \\
(k-\omega_j^2 m)a_{1j} - k a_{2j} &= 0 \qquad (5.42) \\
-k a_{1j} + (2k - \omega_j^2 M) a_{2j} - k a_{3j} &= 0 \qquad (5.43) \\
-k a_{2j} + (k - \omega_j^2 m) a_{3j} &= 0 \qquad (5.44)
\end{aligned}$$

The normalization condition is given by

$$\begin{aligned}
\tilde{\mathbf{A}}\mathbf{T}\mathbf{A} &= 1 \\
\begin{bmatrix} a_{1j} & a_{2j} & a_{3j} \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} &= 1 \\
m a_{1j}^2 + M a_{2j}^2 + m a_{3j}^2 &= 1 \qquad (5.45)
\end{aligned}$$

For $\omega_1 = 0$, from equations 5.42 and 5.44,

$$a_{11} = a_{21} = a_{31} \qquad (5.46)$$

By using equation 5.46 in 5.45,

$$a_{11} = a_{21} = a_{31} = \frac{1}{\sqrt{2m+M}} \qquad (5.47)$$

In the second mode $k - \omega_2^2 m = 0$, from equation 5.42, $a_{22} = 0$, then from equation 5.43, $a_{12} = -a_{32}$. The equation 5.45 gives,

$$a_{12} = \frac{1}{\sqrt{2m}} \quad \text{and} \quad \mathrm{a}_{32} = -\frac{1}{\sqrt{2\mathrm{m}}} \qquad (5.48)$$

For ω_3, from equation 5.42 and 5.44, $a_{13} = a_{33}$. In the third mode $\omega_3^2 = \frac{k}{m}\left(1 + \frac{2m}{M}\right)$, the equation 5.43 becomes,

$$\begin{aligned}
-k a_{13} + \left[2k - \frac{k}{m}\left(1 + \frac{2m}{M}\right)\right] a_{23} - k a_{13} &= 0 \\
a_{23} &= -\frac{2m}{M} a_{13} \qquad (5.49)
\end{aligned}$$

Using $a_{13} = a_{33}$, and equation 5.49 in equation 5.45,

$$\begin{aligned}
2m a_{13}^2 + M\left(-\frac{2m}{M} a_{13}\right)^2 &= 1 \\
a_{13} &= \frac{1}{\sqrt{2m\left(1 + \frac{2m}{M}\right)}} \qquad (5.50)
\end{aligned}$$

Equation 5.50 in equation 5.49,

$$a_{23} = -\frac{2m}{M} \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} = \frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} \qquad (5.51)$$

Then the $\mathbf{A}$ matrix can be written as,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2m+M}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \\ \frac{1}{\sqrt{2m+M}} & 0 & \frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} \\ \frac{1}{\sqrt{2m+M}} & -\frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \end{bmatrix}$$

The $\mathbf{A}^\dagger$ can be evaluated as

$$\mathbf{A}^\dagger = \begin{bmatrix} \frac{m}{\sqrt{2m+M}} & \frac{M}{\sqrt{2m+M}} & \frac{m}{\sqrt{2m+M}} \\ \sqrt{\frac{m}{2}} & 0 & -\sqrt{\frac{m}{2}} \\ \sqrt{\frac{mM}{2(2m+M)}} & -\sqrt{\frac{2mM}{(2m+M)}} & \sqrt{\frac{mM}{2(2m+M)}} \end{bmatrix}$$

The normal coordiantes are given by

$$\xi = \mathbf{A}^\dagger \eta \tag{5.52}$$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \frac{m}{\sqrt{2m+M}} & \frac{M}{\sqrt{2m+M}} & \frac{m}{\sqrt{2m+M}} \\ \sqrt{\frac{m}{2}} & 0 & -\sqrt{\frac{m}{2}} \\ \sqrt{\frac{mM}{2(2m+M)}} & -\sqrt{\frac{2mM}{(2m+M)}} & \sqrt{\frac{mM}{2(2m+M)}} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \tag{5.53}$$

$$\xi_1 = \frac{1}{\sqrt{2m+M}}\left[m(\eta_1+\eta_3) + M\eta_2\right] \tag{5.54}$$

$$\xi_2 = \sqrt{\frac{m}{2}}(\eta_1 - \eta_3) \tag{5.55}$$

$$\xi_3 = \sqrt{\frac{mM}{2m+M}}\left[(\eta_1+\eta_3) - \sqrt{2}\eta_2\right] \tag{5.56}$$

Any general longitudinal vibration of the molecule will be linear combination of the normal modes ω_2 and ω_3. The amplitudes of the normal modes, and their phases relative to each other can be determined by the initial conditions

5.4 Normal modes of double pendulum

From the figure 5.5 we can write

$$\begin{aligned} x_1 &= l\sin\theta_1 & y_1 &= -l\cos\theta_2 \\ \dot{x_1} &= l\cos\theta_1\dot{\theta_1} & \dot{y_1} &= l\sin\theta_1\dot{\theta_1} \\ x_2 &= x_1 + l\sin(\theta_1+\theta_2) & y_2 &= y_1 - l\cos(\theta_1+\theta_2) \\ \dot{x_2} &= \dot{x_1} + l\cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) & \dot{y_2} &= \dot{y_1} + l\sin(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) \\ \dot{x_2} &= l\cos\theta_1\dot{\theta_1} + l\cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) & \dot{y_2} &= l\sin\theta_1\dot{\theta_1} + l\sin(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) \end{aligned}$$

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x_1}^2 + \dot{y_i}^2 + \dot{x_2}^2 + \dot{y_2}^2) \\ &= \frac{1}{2}ml^2[2\dot{\theta_1}^2 + 2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})\cos\theta_2 + (\dot{\theta_1}+\dot{\theta_2})^2] \end{aligned}$$

The potential energy is

$$\begin{aligned} V &= mg(y_1+y_2) \\ &= -mgl[2\cos\theta_1 + \cos(\theta_1+\theta_2)] \\ &= mgl\left[4\sin^2\frac{\theta_1}{2} + 2\sin^2\frac{(\theta_1+\theta_2)}{2}\right] \end{aligned}$$

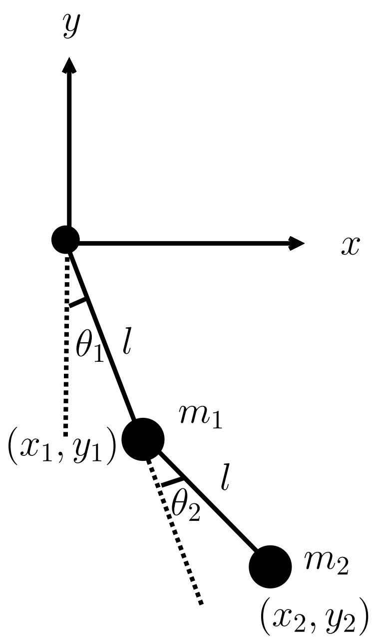

Figure 5.5: Double pendulum.

For small oscillations $\sin\theta \to \theta$ and $\cos\theta \to 1$, then T and V expressions can be written as

$$\begin{aligned}
T &= \frac{1}{2}ml^2[2\dot{\theta}_1^{\,2} + 2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + (\dot{\theta}_1 + \dot{\theta}_2)^2] \\
&= ml^2\left[\frac{5}{2}\dot{\theta}_1^{\,2} + 2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^{\,2}\right] \\
&= ml^2 \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix} \begin{bmatrix} \frac{5}{2} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \\
V &= mgl\left[\theta_1^2 + \frac{(\theta_1 + \theta_2)^2}{2}\right] \\
&= \frac{mgl}{2}\left[3\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2\right] \\
&= \frac{mgl}{2} \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}
\end{aligned}$$

Then

$$[\mathbf{V} - \omega^2\mathbf{T}] = \frac{mgl}{2} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} - \omega^2 \, \frac{ml^2}{2} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = 0$$

On substituting $\omega^2 = \lambda g/l$

$$\begin{aligned}
[\mathbf{V} - \omega^2\mathbf{T}] &= \frac{mgl}{2} \begin{bmatrix} 3-5\lambda & 1-2\lambda \\ 1-2\lambda & 1-\lambda \end{bmatrix} = 0 \\
&= \frac{mgl}{2}(\lambda^2 - 4\lambda + 2) = 0 \\
\lambda &= 2 \pm \sqrt{2} \\
\lambda_1 &= 2 - \sqrt{2}, \qquad \lambda_2 = 2 + \sqrt{2} \\
\omega_1^2 &= (2 - \sqrt{2})\frac{g}{l}, \qquad \omega_2^2 = (2 + \sqrt{2})\frac{g}{l}
\end{aligned}$$

Chapter 6

The Hamilton equations of motion

Lagrange formulation is in terms of generalized coordinates q_i and generalized velocities $\dot{q}_i$ gives equations of motion, which are second order in time. Instead if we regard N generalized coordinates q_i and N generalized momenta p_i as independent variables, and again $q(t)$ and $p(t)$ at every instant of time t, we will get $2N$ first order equations. Hence the $2N$ equations of motion describe the behaviour of the system ina phase space whose coodinates are the 2 independent variables. These are called *canonical coordinates* and *canonical momenta*. This new formulation is by the Hamiltonian and is known as Hamiltonian formulation.
The Lagrange equations for a free particle can be written as

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = 0 \tag{6.1}$$

where

$$\begin{aligned} L(q,\dot{q},t) &= T - V = \frac{1}{2}\sum_i m_i \dot{q}_i{}^2 - V \\ \frac{\delta L}{\delta \dot{q}_i} &= m_i \dot{q}_i = p_i \end{aligned} \tag{6.2}$$

p_i are called generalized or conjugate momenta. Equation 6.2 in 6.1 gives,

$$\begin{aligned} \frac{dp_i}{dt} - \frac{\delta L}{\delta q_i} &= 0 \\ \dot{p}_i &= \frac{\delta L}{\delta q_i} \end{aligned} \tag{6.3}$$

The differential of the Lagrangian can be written as

$$dL = \sum_i \frac{\delta L}{\delta q_i} dq_i + \sum_i \frac{\delta L}{\delta \dot{q}_i} d\dot{q}_i + \frac{\delta L}{\delta t} dt \tag{6.4}$$

Equations 6.2 and 6.3 in 6.4,

$$dL = \sum_i \dot{p}_i \, dq_i + \sum_i p_i \, d\dot{q}_i + \frac{\delta L}{\delta t} \, dt \tag{6.5}$$

If we define the Hamiltonian $H(q,p,t)$ as a function of generalized coordinates q_i and generalized momenta p_i, the Legendre transformation generate the Hamiltonian

$$H(q,p,t) = \sum_i \dot{q}_i p_i - L(q,\dot{q},t) \tag{6.6}$$

The differential of the Hamiltonian is

$$\sum_i \frac{\delta H}{\delta q_i} dq_i + \sum_i \frac{\delta H}{\delta p_i} dp_i + \frac{\delta H}{\delta t} dt = \sum_i \dot{q}_i \, dp_i + \sum_i p_i \, d\dot{q}_i - dL \tag{6.7}$$

Equation 6.5 in 6.7,

$$\sum_i \frac{\delta H}{\delta q_i} dq_i + \sum_i \frac{\delta H}{\delta p_i} dp_i + \frac{\delta H}{\delta t} dt = \sum_i \dot{q}_i \, dp_i - \sum_i \dot{p}_i \, dq_i - \frac{\delta L}{\delta t} dt$$

$$\dot{q}_i = \frac{\delta H}{\delta p_i} \tag{6.8}$$

$$-\dot{p}_i = \frac{\delta H}{\delta q_i} \tag{6.9}$$

$$-\frac{\delta L}{\delta t} = \frac{\delta H}{\delta t} \tag{6.10}$$

Equations 6.8 and 6.9 are known as the canonical equations of Hamilton. They constitute the desired set of $2N$ first order equations of motion replacing the N second order Lagrange equations.

If (x, y, z) are the Cartesian coordinates at time t of a free material point of mass m moving in a potential field $V(x, y, z) = V(q_i)$, we may take $q_1 = x, q_2 = y, q_3 = z$. The kinetic energy T is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\sum_i \dot{q_i}^2$$

The Lagrangian for the particle is

$$\begin{aligned} T - V = L &= \frac{1}{2}m\sum_i \dot{q_i}^2 - V(q_i) \\ \frac{\delta L}{\delta \dot{q_i}} &= m\dot{q_i} \\ p_i &= m\dot{q_i} \end{aligned}$$

On substituting for L and p_i in the equation 6.6,

$$\begin{aligned} H &= \sum_i \dot{q_i}p_i - L = m\sum_i \dot{q_i}^2 - (T - V) \\ H &= T + V \end{aligned} \tag{6.11}$$

Thus the Hamiltonian becomes the total energy of the system.

6.1 Examples of the Hamiltonian

6.1.1 Hamiltonian for a free particle in different coordinates

1. *Using Cartesian coordinates:* (x, y, z) are the Cartesian coordinates at time t of a free material point of mass m moving in a potential field $V(x, y, z)$. The kinetic energy T is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus the Hamiltonian for the particle is

$$\begin{aligned} T + V = H &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z) \\ H &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \end{aligned} \tag{6.12}$$

2. *Using cylindrical polar coordinates*: (r, θ, z) are the cylindrical coordinates at time t of a free material point of mass m in the potential field $V(r)$.
The kinetic energy T is

$$\begin{aligned} T &= \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2\right] \\ &= \frac{1}{2m}\left[(m\dot{r})^2 + \frac{1}{r^2}(mr^2\dot{\theta})^2 + (m\dot{z})^2\right] \\ T &= \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right] \end{aligned}$$

Then

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right] + V(r)$$

3. *Using spherical polar coordinates*: (r, θ, ϕ) are the spherical polar coordinates at time t of a free material point of mass m in the potential field $V(r)$.
The kinetic energy T is

$$\begin{aligned} T &= \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \left(rsin\theta\dot{\phi}\right)^2\right] \\ &= \frac{1}{2m}\left[(m\dot{r})^2 + \frac{1}{r^2}\left(r^2\dot{\theta}\right)^2 + \frac{1}{r^2sin^2\theta}\left(r^2sin^2\theta\dot{\phi}\right)^2\right] \\ T &= \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2sin^2\theta}\right] \end{aligned}$$

Then

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right] + V(r)$$

6.1.2 Hamiltonian for an electron in a Coulomb field

When an electron revolving about the charge e,

$$v(r) = -\frac{e^2}{r}$$

The kinetic energy T of electron in spherical coordinate is

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right]$$

Then

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right] - \frac{e^2}{r}$$

6.1.3 Hamiltonian for the simple harmonic oscillator

The Lagrangian for a simple harmonic oscillator can be written as

$$L = \frac{1}{2}m\sum_i \dot{q_i}^2 - \frac{1}{2}m\omega^2\sum_i {q_i}^2 = \sum_i \frac{p_i^2}{2m} - \frac{1}{2}m\omega^2\sum_i {q_i}^2$$

The cononical momentum is

$$\begin{aligned} p_i &= \frac{\delta L}{\delta \dot{q}} = m\dot{q}_i \\ \dot{q}_i &= \frac{p_i}{m} \end{aligned}$$

Then

$$\begin{aligned} H &= \sum_i p_i\dot{q}_i - L = \sum_i \frac{p_i^2}{m} - \sum_i \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2\sum_i {q_i}^2 \\ H &= \sum_i \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2\sum_i {q_i}^2 \end{aligned}$$

6.1.4 Hamiltonian for an electron in electromagnetic field

Consider a particle of mass m and charge e moving in an electromagnetic field. Lagrangian for the particle is

$$L = T - U = \frac{1}{2}m\sum_i \dot{q_i}^2 - e\left(\phi - \vec{A}.\sum_i \dot{q}_i\right)$$

where $e\left(\phi - \vec{A}.\dot{q}_i\right)$ is the velocity dependent potential.

$$\begin{aligned} \frac{\delta L}{\delta \dot{q}_i} = p_i &= m\dot{q}_i + e\vec{A} \\ \dot{q}_i &= \frac{1}{m}(p_i - e\vec{A}) \end{aligned}$$

The Hamiltonian H is

$$
\begin{aligned}
H &= \sum_i p_i \dot{q}_i - L \\
&= m\sum_i \dot{q_i}^2 + e\vec{A}.\sum_i \dot{q}_i - \frac{1}{2}m\sum_i \dot{q_i}^2 + e\left(\phi - \vec{A}.\sum_i \dot{q}_i\right) \\
&= \frac{1}{2}m\sum_i \dot{q_i}^2 + e\phi \\
&= \frac{1}{2m}\left(\sum_i p_i - e\vec{A}\right)^2 + e\phi \\
H &= \frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\phi
\end{aligned}
$$

6.2 Cyclic coordinates

Consider a system of N degrees of freedom described by q_i generalized coordinates. The Lagrange's equations for the system are

$$
\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = 0
$$

If Lagrangian of the system does not contain a given coordinate q_i even though it may contain corresponding velocity $\dot{q}_i$, then the coordinate q_i is said to be *cyclic or ignorable.* Then

$$
\frac{\delta L}{\delta q_i} = 0
$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) &= 0 \\
\frac{dp_i}{dt} &= 0 \\
p_i &= constant
\end{aligned}
$$

The generalized momentum conjugate to a cyclic coordinate is conserved.
Example: In a planetory motion, the angular momentum p_θ is constant
$p_\theta = mr^2\dot{\theta} = constant$. Here θ is cyclic.

6.3 Hamilton's Equations from a Variational Principle

The motion of a conservative system from its configuration at time t_1 to its configuration at time t_2 is such that the line integral between the time t_1 and t_2 of the Lagrangian of the system has a stationary value for the actual path of the motion.

$$
I = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = constant
$$

$L = T - V$ is the Lagrangian. Since $\int L\,dt$ has the dimensions of $energy \times time$ called action, the principle is sometimes referred to as the principle of least action. The integral is called the action integral.
The variation of the action integral for fixed time t_1 and t_2 must be zero.

$$
\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\;\; dt = 0
$$

Byusing equation 6.6,

$$
\begin{aligned}
\delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\;\; dt &= \delta \int_{t_1}^{t_2} \left(\sum_i \dot{q}_i p_i - H\right) dt = 0 \\
&= \int_{t_1}^{t_2} \left(\sum_i \dot{q}_i \delta p_i + \sum_i p_i \delta \dot{q}_i - \delta H\right) dt = 0 \qquad (6.13)
\end{aligned}
$$

Since,

$$\delta H(q,p) = \sum_i \frac{\delta H}{\delta q_i}\delta q_i + \sum_i \frac{\delta H}{\delta p_i}\delta p_i \tag{6.14}$$

Equation 6.14 in equation 6.13,

$$\sum_i \int_{t_1}^{t_2} \left(\dot{q}_i - \frac{\delta H}{\delta p_i}\right)\delta p_i\,dt + \int_{t_1}^{t_2} p_i \delta\dot{q}_i\,dt - \int_{t_1}^{t_2} \frac{\delta H}{\delta q_i}\delta q_i\,dt = 0$$

$$\sum_i \int_{t_1}^{t_2} \left(\dot{q}_i - \frac{\delta H}{\delta p_i}\right)\delta p_i\,dt + p_i\,\delta q_i|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i\,\dot{p}_i\,dt - \int_{t_1}^{t_2} \frac{\delta H}{\delta q_i}\delta q_i\,dt = 0$$

Since the variation $\delta q_i = 0$ at the end point, the term $p_i\,dq_i|_{t_1}^{t_2} = 0$.

$$\sum_i \int_{t_1}^{t_2} \left(\dot{q}_i - \frac{\delta H}{\delta p_i}\right)\delta p_i\,dt - \int_{t_1}^{t_2} \left(\dot{p}_i + \frac{\delta H}{\delta q_i}\right)\delta q_i\,dt = 0$$

$$\sum_i \int_{t_1}^{t_2} \left[\left(\dot{q}_i - \frac{\delta H}{\delta p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\delta H}{\delta q_i}\right)\delta q_i\right] dt = 0$$

Since the system is holonomic and is described in the phase space, q_i's and p_i's are all independent, and δq_i's and δp_i's are arbitrary at all points of the path. The above integrals can vanish, only if

$$\begin{aligned} \dot{q}_i - \frac{\delta H}{\delta p_i} &= 0 \qquad \Longrightarrow \qquad \dot{q}_i = \frac{\delta H}{\delta p_i} \\ \dot{p}_i + \frac{\delta H}{\delta q_i} &= 0 \qquad \Longrightarrow \qquad -\dot{p}_i = \frac{\delta H}{\delta q_i} \end{aligned}$$

which are Hamilton's eqiuations of motion.

6.4 Canonical Transformations

Canonical transformations are transformations of the coordinates and momenta (q,p) that preserve Hamilton's equations (though with a different Hamiltonian).
The transformations from one set of coordinates q_i to a new set Q_i, by transformation equations of the form

$$Q_i = Q_i(q,t) \tag{6.15}$$

are called *point transformations.* It can be shown that under a point transformation, a system that obeys the Euler-Lagrange equations in the original coordinates continues to obey them in the new coordinates.
In the Hamiltonian formulation the momenta are also independent variables on the same level as the generalized coordinates. The concept of transformation of coordinates must therefore be widened to include the simultaneous transformation of the independent coordinates and momenta, (q_i,p_i), to a new set Q_i, P_i, with equations of transformation

$$Q_i = Q_i(q,p,t) \tag{6.16}$$
$$P_i = P_i(q,p,t) \tag{6.17}$$

These transformations are called *contact transformations.* An arbitrary contact transformation may not preserve Hamilton's equations. The transformations which preserve Hamilton's equations are known as *canonical transformations.*
Equations 6.15 define a *point transformation of configuration space* and, equations 6.16 and 6.17 define a *point transformation of phase space.*
$H(Q,P,t)$ is the Hamiltonian in the cananical coordinates and the equations of motion in the new cocodinates are in the Hamiltonian form

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} \qquad \dot{P}_i = -\frac{\delta H}{\delta Q_i} \tag{6.18}$$

The Hamilton principle in both old coordinates (q_i,p_i) and canonical coordinates (Q_i,P_i) are written as

$$\delta \int_{t_1}^{t_2} \left(\sum_i \dot{q}_i p_i - H(q,p,t)\right) dt = 0 \tag{6.19}$$

$$\delta \int_{t_1}^{t_2} \left(\sum_i \dot{Q}_i P_i - H(Q,P,t)\right) dt = 0 \tag{6.20}$$

The simultaneous validity of equations 6.19 and 6.20 does not mean that the integrands in both expressions are equal. Since the general form of the modified Hamilton's principle has zero variation at the end points, the equations 6.19 and 6.20 will be satisfied if the integrands are connected by a relation of the form

$$\lambda\left[\dot{q}_i p_i - H(q,p,t)\right] = \dot{Q}_i P_i - H(Q,P,t) + \frac{dF}{dt} \tag{6.21}$$

Here F is any function of the phase space coordinates with continuous second derivatives called *generating function*, and λ is a constant known as a *scale transformation.* For canonical transformationc $\lambda = 1$ and the transformation for which $\lambda \neq 1$ is called *extended canonical transformation.*

The term $\frac{dF}{dt}$ in equation 6.21 contributes to the variation of the action integral only at the end points and will therefore vanish if F is a function of (q, p, t) or (Q, P, t) or any mixture of the phase space coordinates. F is useful for specifying the exact form of the canonical transformation only when half of the variables are from the old set and half are from the new set.

If $F = F_1(q, Q, t)$, the equation 6.21 (with $\lambda = 1$) becomes

$$\dot{q}_i p_i - H(q,p,t) = \dot{Q}_i P_i - H(Q,P,t) + \frac{\delta F_1}{\delta t} + \frac{\delta F_1}{\delta q_i}\dot{q}_i + \frac{\delta F_1}{\delta Q_i}\dot{Q}_i$$

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_1}{\delta t} + \left(\frac{\delta F_1}{\delta q_i} - p_i\right)\dot{q}_i + \left(\frac{\delta F_1}{\delta Q_i} + P_i\right)\dot{Q}_i \tag{6.22}$$

Since the old and the new coordinates, q_i, and Q_i, are separately independent, equation 6.22 can hold identically only if the coefficients of $\dot{q}_i$, and $\dot{Q}_i$ each vanish. Thus

$$p_i = \frac{\delta F_1}{\delta q_i} \tag{6.23}$$

$$P_i = -\frac{\delta F_1}{\delta Q_i} \tag{6.24}$$

Equations 6.23 and 6.24 in equation 6.22,

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_1}{\delta t} \tag{6.25}$$

The function $F(q, Q, t)$ is the generating function of the cananical transformation and it specifies the required equations of the transformation.

If $F = F_2(q, P, t) - Q_i P_i$, the equation 6.21 (with $\lambda = 1$) becomes

$$\dot{q}_i p_i - H(q,p,t) = \dot{Q}_i P_i - H(Q,P,t) + \frac{\delta F_2}{\delta t} + \frac{\delta F_2}{\delta q_i}\dot{q}_i + \frac{\delta F_2}{\delta P_i}\dot{P}_i - Q_i\dot{P}_i - \dot{Q}_i P_i$$

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_2}{\delta t} + \left(\frac{\delta F_2}{\delta q_i} - p_i\right)\dot{q}_i + \left(\frac{\delta F_2}{\delta P_i} - Q_i\right)\dot{P}_i \tag{6.26}$$

Since the old and the new coordinates, q_i, and Q_i, are separately independent, equation 6.22 can hold identically only if the coefficients of $\dot{q}_i$, and $\dot{Q}_i$ each vanish. Thus

$$p_i = \frac{\delta F_2}{\delta q_i} \tag{6.27}$$

$$Q_i = \frac{\delta F_2}{\delta P_i} \tag{6.28}$$

Equations 6.27 and 6.28 in equation 6.26,

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_2}{\delta t} \tag{6.29}$$

6.4.1 Other forms of the Generating Function

We made a particular choice to make F a function of q_i and Q_i. Given the symmetry between coordinates and canonical momenta, it is likely that we could equally well write F as a function of $(q_i, P_i), (p_i, P_i)$ or (p_i, Q_i). These different generating functions are simply different ways to generate the same canonical transformation $(q_i, p_i) \rightarrow (Q_i, P_i)$. The four basic cananical transfornmations are given in the table 6.4.1.

Consider old and new coordinates,

$$q_i = q_i(Q,P)$$
$$p_i = p_i(Q,P)$$
$$P_i = P_i(q,p)$$
$$Q_i = Q_i(q,p)$$

Generating function	Generating function derivatives
$F = F_1(q, Q, t)$	$p_i = \frac{\delta F_1}{\delta q_i} \qquad P_i = -\frac{\delta F_1}{\delta Q_i}$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\delta F_2}{\delta q_i} \qquad Q_i = \frac{\delta F_2}{\delta P_i}$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\delta F_3}{\delta p_i} \qquad P_i = -\frac{\delta F_3}{\delta Q_i}$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$p_i = -\frac{\delta F_4}{\delta p_i} \qquad Q_i = \frac{\delta F_4}{\delta P_i}$

Table 6.1: Different forms of the Generating Function and their derivatives

$$\begin{aligned} \frac{dQ_i}{dt} &= \frac{\delta Q_i}{\delta q_j}\frac{\delta q_j}{\delta t} + \frac{\delta Q_i}{\delta p_j}\frac{\delta p_j}{\delta t} \\ \frac{\delta H}{\delta P_i} = \dot{Q}_i &= \frac{\delta Q_i}{\delta q_j}\dot{q}_j + \frac{\delta Q_i}{\delta p_j}\dot{p}_j \end{aligned}$$

Since,

$$\begin{aligned} H &= H(Q, P) \\ \frac{\delta H}{\delta P_i} &= \frac{\delta H}{\delta p_j}\frac{\delta p_j}{\delta P_i} + \frac{\delta H}{\delta q_j}\frac{\delta q_j}{\delta P_i} \\ &= \dot{q}_j\frac{\delta p_j}{\delta P_i} - \dot{p}_j\frac{\delta q_j}{\delta P_i} \end{aligned}$$

Therefore,

$$\dot{q}_j\frac{\delta p_j}{\delta P_i} - \dot{p}_j\frac{\delta q_j}{\delta P_i} = \frac{\delta Q_i}{\delta q_j}\dot{q}_j + \frac{\delta Q_i}{\delta p_j}\dot{p}_j$$

That is, the transformation is canonical only if,

$$\left(\frac{\delta p_j}{\delta P_i}\right)_{Q,P} = \left(\frac{\delta Q_i}{\delta q_j}\right)_{q,p}, \qquad \left(\frac{\delta q_i}{\delta P_i}\right)_{Q,P} = \left(\frac{\delta Q_i}{\delta p_j}\right)_{q,p} \tag{6.30}$$

6.4.2 Examples of canonical transformation

Generating function of the second type

If $F_2 = q_i P_i$

$$\begin{aligned} \frac{\delta F_2}{\delta q_i} &= P_i \\ \frac{\delta F_2}{\delta P_i} &= q_i \end{aligned} \tag{6.31}$$

On comparing the values with the table 6.4.1.,

$$\begin{aligned} \frac{\delta F_2}{\delta q_i} &= P_i = p_i \\ \frac{\delta F_2}{\delta P_i} &= q_i = Q_i \end{aligned} \tag{6.32}$$

Hence $H(q, p, t) = H(Q, P, t)$ and F_2 generates the identity transformation.

Generating function of the first type

If $F_1 = q_i Q_i$

$$\begin{aligned} \frac{\delta F_1}{\delta q_i} &= Q_i \\ \frac{\delta F_1}{\delta Q_i} &= q_i \end{aligned} \tag{6.33}$$

On comparing the values with the table 6.4.1.,

$$\begin{aligned} \frac{\delta F_1}{\delta q_i} &= Q_i = p_i \\ \frac{\delta F_1}{\delta Q_i} &= q_i = -P_i \end{aligned} \tag{6.34}$$

Thus the transformation interchanges the momenta and the coordinates.

Simple harmonic oscillator

The Hamiltonian for a simple harmonic oscillator can be written as

$$H(q_i, p_i) = \frac{1}{2m}\left(\sum_i p_i^2 + m^2\omega^2 \sum_i q_i^2\right) \tag{6.35}$$

This form of the Hamiltonian, as the sum of two squares, suggests a transformation in which $H(q, p, t)$ is cyclic in the new coordinate. Then a canonical transformation takes the form

$$p_i = f(P)\, cos\, Q_i \tag{6.36}$$

$$q_i = \frac{f(P)}{m\omega} sin\, Q_i \tag{6.37}$$

Substituting for p_i^2 and q_i^2 by using equations 6.36 and 6.37 to equation 6.35,

$$H(Q_i, P_i) = \frac{1}{2m}\left(\sum_i f(P)^2 cos^2 Q_i + m^2\omega^2 \sum_i \frac{f(P)^2}{m^2\omega^2} sin^2 Q_i\right)$$

$$H(Q_i, P_i) = \frac{f(P)^2}{2m} \tag{6.38}$$

If we use a generating function given by

$$F_1 = \frac{m\omega q_i^2}{2} cot\, Q_i \tag{6.39}$$

Then

$$p_i = \frac{\delta F_1}{\delta q_i} = m\omega q_i\, cot\, Q_i$$

$$P_i = -\frac{\delta F_1}{\delta Q_i} = \frac{m\omega q_i^2}{2sin^2 Q_i}$$

$$q_i = \sqrt{\frac{2P_i}{m\omega}} sin\, Q_i \tag{6.40}$$

$$p_i = \sqrt{2P_i m\omega}\, cos\, Q_i \tag{6.41}$$

On comparing equations 6.40 and 6.41 with equations 6.36 and 6.37,

$$f(P_i) = \sqrt{2P_i m\omega} \tag{6.42}$$

Equation 6.42 in equation 6.38 gives

$$H = \omega \sum_i P_i = \sum_i E_i \tag{6.43}$$

$$\sum_i P_i = \frac{\sum_i E_i}{\omega}$$

where E_i is the total energy of a oscillator. The equations of motion in the cananical coordinates is

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} = \omega \quad (6.44)$$

$$Q_i = \omega t + \alpha \quad (6.45)$$

where α is the constant of integration evaluated by the initial conditions.
From equations 6.40 and 6.41, the solutions for p and q are written as

$$q_i = \sqrt{\frac{2E_i}{m\omega^2}} sin\,(\omega t + \alpha) \quad (6.46)$$

$$p_i = \sqrt{2mE_i}\, cos\,(\omega t + \alpha) \quad (6.47)$$

From equation 6.40 and 6.41, it can see that we have transformed from simple position q_i and momentum p_i to phase Q_i and energy P_i of the oscillatory motion. Equation 6.46 shows that the energy depends only on the oscillator amplitude. This kind of transformation is going to have obvious use when dealing with mechanical or electromagnetic waves.

6.5 Poisson brackets

Poisson brackets are a powerful and sophisticated tool in the Hamiltonian formalism of Classical Mechanics. They also happen to provide a direct link between classical and quantum mechanics. A classical system with N degrees of freedom, say a set of $N/3$ particles in three dimensions, is described by $2N$ phase space coordinates. These are the N generalized coordinates $q_1, q_2, q_3, ..., q_N$ and N conjugate momenta $p_1, p_2, p_3, ..., p_N$. The Hamiltonian of the system depends on these $2N$ variables and possibly on time t as well, and it can be expressed as

$$H(q_1, q_2, q_3, ..., q_N, p_1, p_2, p_3, ..., p_N, t) = H(q_i, p_i, t)$$

The Poisson bracket is an operation which takes two functions of phase space and time, call them $F(q_i, p_i, t)$ and $G(q_i, p_i, t)$ and produces a new function. With respect to cananical coordinates (q_i, p_i), it is defined as

$$[F, G] = \sum_i^N \left(\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta G}{\delta q_i} \right) \quad (6.48)$$

In the case of a single degree of freedom, $N = 1$, phase space is 2-dimensional, (q, p) and the Poisson bracket has only two terms

$$[F, G] = \left(\frac{\delta F}{\delta q}\frac{\delta G}{\delta p} - \frac{\delta F}{\delta p}\frac{\delta G}{\delta q} \right)$$

Time derivative of the function $F(q_i, p_i, t)$ is

$$\frac{dF}{dt} = \sum_i^N \left(\frac{\delta F}{\delta q_i}\frac{\delta q_i}{\delta t} + \frac{\delta F}{\delta p_i}\frac{\delta p_i}{\delta t} \right) + \frac{\delta F}{\delta t}$$

$$\frac{dF}{dt} = \sum_i^N \left(\frac{\delta F}{\delta q_i}\dot{q}_i + \frac{\delta F}{\delta p_i}\dot{p}_i \right) + \frac{\delta F}{\delta t} \quad (6.49)$$

By using the Hamiltonian equations of motion

$$\dot{q}_i = \frac{\delta H}{\delta p_i} \qquad \dot{p}_i = -\frac{\delta H}{\delta q_i}$$

equation 6.49 becomes,

$$\frac{dF}{dt} = \sum_i^N \left(\frac{\delta F}{\delta q_i}\frac{\delta H}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta H}{\delta q_i} \right) + \frac{\delta F}{\delta t} \quad (6.50)$$

$$\frac{dF}{dt} = [F, H] + \frac{\delta F}{\delta t} \quad (6.51)$$

Equation 6.50 is the equation of motion of the function F expressed in terms of Poisson bracket.

6.5.1 Properties of Poisson bracket

1. Consider

$$
\begin{aligned}
[F, G_1 + G_2] &= [F, G_1] + [F, G_2] \\
[F, G_1 G_2] &= [F, G_1]G_2 + G_1[F, G_2]
\end{aligned}
$$

2. The Poisson bracket is anti-symmetric in its two arguments

$$[G, F] = -[F, G]$$

An immediate consequence of this is that $[F, F] = 0$ for any function at all.

$$
\begin{aligned}
[q_j, q_k] &= 0 = [p_j, p_k] \\
[q_j, p_k] &= \delta_{jk} = -[p_j, q_k]
\end{aligned}
$$

3. The Poisson bracket is linear in either of its arguments

$$
\begin{aligned}
[F_1 + F_2, G] &= [F_1, G] + [F_2, G] \\
[F, G_1 + G_2] &= [F, G_1] + [F, G_2] \\
[F, G_1 G_2] &= [F, G_1]G_2 + G_1[F, G_2]
\end{aligned}
$$

4. **Invariance under canonical transformations:**

$$
\begin{aligned}
[F, G]_{(q,p)} &= \sum_i^N \left(\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta G}{\delta q_i} \right) \\
&= \left(\frac{\delta F}{\delta q_1}\frac{\delta G}{\delta p_1} - \frac{\delta F}{\delta p_1}\frac{\delta G}{\delta q_1} \right) + \left(\frac{\delta F}{\delta q_2}\frac{\delta G}{\delta p_2} - \frac{\delta F}{\delta p_2}\frac{\delta G}{\delta q_2} \right) + \ldots\ldots\ldots\ldots \\
&\quad + \left(\frac{\delta F}{\delta q_n}\frac{\delta G}{\delta p_n} - \frac{\delta F}{\delta p_n}\frac{\delta G}{\delta q_n} \right)
\end{aligned}
$$

$$
= \begin{bmatrix} \frac{\delta F}{\delta q_1} & \frac{\delta F}{\delta q_2}\cdots & \frac{\delta F}{\delta q_n} & \frac{\delta F}{\delta p_1} & \frac{\delta F}{\delta p_2}\cdots & \frac{\delta F}{\delta p_n} \end{bmatrix}
\begin{bmatrix} 0 & 0.... & 1 & 0 \\ 0 & 0..... & 0 & 1 \\ . & & & \\ . & & & \\ . & & & \\ -1 & 0.... & 0 & 0 \\ 0 & -1..... & 0 & 0 \end{bmatrix}
\begin{bmatrix} \frac{\delta G}{\delta q_1} \\ \frac{\delta G}{\delta q_2} \\ . \\ . \\ \frac{\delta G}{\delta q_n} \\ \frac{\delta G}{\delta p_1} \\ \frac{\delta G}{\delta p_2} \\ . \\ . \\ \frac{\delta G}{\delta p_n} \end{bmatrix}
$$

$$[F, G]_\eta = \widetilde{\left[\frac{\delta F}{\delta \eta}\right]} \mathbf{J} \left[\frac{\delta G}{\delta \eta}\right]$$

where η consists of $2n$ elements of (q_i, p_i) and $\mathbf{J}$ is the $2n \times 2n$ anti-symmetric matrix. For $i = 2$, the above equation can be written as

$$
[F, G]_{(q,p)} = \begin{bmatrix} \frac{\delta F}{\delta q_1} & \frac{\delta F}{\delta q_2} & \frac{\delta F}{\delta p_1} & \frac{\delta F}{\delta p_2} \end{bmatrix}
\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \frac{\delta G}{\delta q_1} \\ \frac{\delta G}{\delta q_2} \\ \frac{\delta G}{\delta p_1} \\ \frac{\delta G}{\delta p_2} \end{bmatrix}
$$

η are the old coordinates (q_i, p_i) and $\xi = \xi(\eta)$ are the transformed coordinates (Q_i, P_i), then

$$\begin{aligned}
\frac{\delta F}{\delta \eta_{\mathbf{i}}} &= \frac{\delta F}{\delta \xi_j}\frac{\delta \xi_j}{\delta \eta_i} \\
&= \frac{\delta F}{\delta \xi_1}\frac{\delta \xi_1}{\delta \eta_i} + \frac{\delta F}{\delta \xi_2}\frac{\delta \xi_2}{\delta \eta_i} + ... + \frac{\delta F}{\delta \xi_n}\frac{\delta \xi_n}{\delta \eta_i} \\
&= \begin{bmatrix} \frac{\delta \xi_1}{\delta \eta_i} & \frac{\delta \xi_2}{\delta \eta_i} ...... \frac{\delta \xi_n}{\delta \eta_i} \end{bmatrix} \begin{bmatrix} \frac{\delta F}{\delta \xi_1} \\ \frac{\delta F}{\delta \xi_2} \\ . \\ . \\ \frac{\delta F}{\delta \xi_n} \end{bmatrix} \\
\left[\frac{\delta F}{\delta \eta}\right] &= \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]}\left[\frac{\delta F}{\delta \xi}\right]
\end{aligned}$$

Similarly,

$$\left[\frac{\delta G}{\delta \eta}\right] = \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]}\left[\frac{\delta G}{\delta \xi}\right]$$

Therefore,

$$\begin{aligned}
[F, G]_\eta &= \widetilde{\widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]}\left[\frac{\delta F}{\delta \xi}\right]} \mathbf{J} \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]}\left[\frac{\delta G}{\delta \xi}\right] \\
&= \widetilde{\left[\frac{\delta F}{\delta \xi}\right]}\left[\frac{\delta \xi}{\delta \eta}\right] \mathbf{J} \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]}\left[\frac{\delta G}{\delta \xi}\right]
\end{aligned}$$

If the transformation $\eta \longrightarrow \xi$ is canonical,

$$\left[\frac{\delta \xi}{\delta \eta}\right] \mathbf{J} \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} = \mathbf{J}$$

Therefore,

$$\begin{aligned}
[F, G]_\eta &= \widetilde{\left[\frac{\delta F}{\delta \xi}\right]} \mathbf{J} \left[\frac{\delta G}{\delta \xi}\right] \\
&= [F, G]_\xi \\
[F, G]_{(q,p)} &= [F, G]_{(Q,P)}
\end{aligned}$$

6.5.2 Constants of Motion

A constant of the motion is some function of phase space, independent of time, $F(q_i, p_i)$, whose value is constant for any particle. In other words, $F(q_i, p_i)$ is a constant of the motion if

$$\frac{dF}{dt} = 0.$$

Since we specified that F does not depend explicitly on time, it follows that

$$\frac{\delta F}{\delta t} = 0$$

Then from equation 6.51,

$$[F, H] = 0$$

Thus F is a constant of the motion if and only if $[F, H] = 0$ for all points in phase space.

- **Energy:** Due to the anti-symmetry of the Poisson bracket $[H, H] = 0$. Using this in equation 6.51,

$$\frac{dH}{dt} = \frac{\delta H}{\delta t} \tag{6.52}$$

If the Hamiltonian does not depend on time explicitly,

$$\frac{\delta H}{\delta t} = 0$$

Then

$$\frac{dH}{dt} = 0, \qquad H(q_i, p_i) = constant \tag{6.53}$$

That is *energy is conserved in cases where the Hamiltonian is time-independent.*

- **Linear Momentum:** In a case where the Hamiltonian does not contain a particular coordinate, q_i , explicitly it is said to be cyclic in that coordinate. Then

$$[p_i, H] = \frac{\delta p_i}{\delta q_i}\frac{\delta H}{\delta p_i} - \frac{\delta p_i}{\delta p_i}\frac{\delta H}{\delta q_i} = -\frac{\delta H}{\delta q_i} \qquad \left(\frac{\delta p_i}{\delta q_i} = 0\right)$$

Since q_i is cyclic, $(\delta H/\delta q_i) = 0$, then $[p_i, H] = 0$, so p_i is a constant of the motion. Thus the *momentum is conserved if it is conjugate to a cyclic coordinate.*

- **Angular Momentum:** Consider a particle in three dimension, (x, y, z), subject to a central force potential $V(r) = V(x, y, z)$. The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \\ \frac{\delta T}{\delta p_x} &= \frac{p_x}{m}, \qquad \frac{\delta T}{\delta p_y} = \frac{p_y}{m}, \qquad \frac{\delta T}{\delta p_z} = \frac{p_z}{m} \end{aligned} \tag{6.54}$$

The potential energy of the system is

$$V = V(r) = V\left(\sqrt{x^2 + y^2 + z^2}\right)$$

$$\begin{aligned} \frac{\delta V}{\delta x} &= \frac{x}{\sqrt{x^2 + y^2 + z^2}}V'(r), \qquad \frac{\delta V}{\delta y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}V'(r), \\ \frac{\delta V}{\delta z} &= \frac{z}{\sqrt{x^2 + y^2 + z^2}}V'(r) \end{aligned} \tag{6.55}$$

where $V'(r)$ is the potential function. The Hamiltonian of the system is

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \tag{6.56}$$

Angular momentum of the system is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

and the components of L can be written as

$$\begin{aligned} L_z &= xp_y - yp_x \\ L_y &= zp_x - xp_z \\ L_x &= yp_z - zp_x \end{aligned}$$

Consider

$$\begin{aligned} [L_z, H] &= [L_z, T + V] \\ [L_z, H] &= [L_z, T] + [L_z, V] \end{aligned}$$

$$
\begin{aligned}
[L_z, T] &= \frac{\delta L_z}{\delta q_i}\frac{\delta T}{\delta p_i} - \frac{\delta L_z}{\delta p_i}\frac{\delta T}{\delta q_i} \\
&= \frac{\delta L_z}{\delta x}\frac{\delta T}{\delta p_x} - \frac{\delta L_z}{\delta p_x}\frac{\delta T}{\delta x} + \frac{\delta L_z}{\delta y}\frac{\delta T}{\delta p_y} - \frac{\delta L_z}{\delta p_y}\frac{\delta T}{\delta y} \\
&= \frac{\delta L_z}{\delta x}\frac{\delta T}{\delta p_x} + \frac{\delta L_z}{\delta y}\frac{\delta T}{\delta p_y} \qquad \left(\frac{\delta T}{\delta q_i} = 0\right) \\
&= \frac{\delta}{\delta x}(xp_y - yp_x)\,\frac{\delta T}{\delta p_x} + \frac{\delta}{\delta y}(xp_y - yp_x)\frac{\delta T}{\delta p_y}
\end{aligned} \tag{6.57}
$$

By using equation 6.54,

$$
[L_z, T] = p_y\frac{p_x}{m} - p_x\frac{p_y}{m} = 0 \tag{6.58}
$$

$$
\begin{aligned}
[L_z, V] &= \frac{\delta L_z}{\delta q_i}\frac{\delta V}{\delta p_i} - \frac{\delta L_z}{\delta p_i}\frac{\delta V}{\delta q_i} \\
&= \frac{\delta L_z}{\delta x}\frac{\delta V}{\delta p_x} - \frac{\delta L_z}{\delta p_x}\frac{\delta V}{\delta x} + \frac{\delta L_z}{\delta y}\frac{\delta V}{\delta p_y} - \frac{\delta L_z}{\delta p_y}\frac{\delta V}{\delta y} \\
&= -\frac{\delta L_z}{\delta p_x}\frac{\delta V}{\delta x} - \frac{\delta L_z}{\delta p_y}\frac{\delta V}{\delta y} \qquad \left(\frac{\delta V}{\delta p_i} = 0\right) \\
&= -\frac{\delta}{\delta p_x}(xp_y - yp_x)\,\frac{\delta V}{\delta x} - \frac{\delta}{\delta p_y}(xp_y - yp_x)\frac{\delta V}{\delta y} \\
[L_z, V] &= y\frac{\delta V}{\delta x} - x\frac{\delta V}{\delta y}
\end{aligned}
$$

By using equation 6.55,

$$
[L_z, V] = y\frac{x}{\sqrt{x^2 + y^2 + z^2}}V'(r) - x\frac{y}{\sqrt{x^2 + y^2 + z^2}}V'(r) = 0 \tag{6.59}
$$

Equations 6.58 and 6.58 gives

$$
[L_z, H] = [L_z, T] + [L_z, V] = 0
$$

Similarly we can show that $[L_x, H] = 0$, $[L_y, H] = 0$. Therefore for *a particle moving in a central force potential all three components of angular momentum are conserved.*

6.5.3 Angular momentum and Poisson bracket relations

Angular momentum of the system is defined as

$$
\vec{L} = \vec{r} \times \vec{p}
$$

The components of L in Cartesian coordinates are

$$
\begin{aligned}
L_z &= xp_y - yp_x \\
L_y &= zp_x - xp_z \\
L_x &= yp_z - zp_x
\end{aligned}
$$

If F is a vector rotating about z axis, the equation of motion in terms of Poisson bracket is

$$
\frac{dF}{d\theta} = [L_z, F] \tag{6.60}
$$

Also we can write

$$
\frac{d\vec{F}}{d\theta} = \hat{k} \times \vec{F} \tag{6.61}
$$

By using equations 6.60 and 6.61,

$$
[L_z, F] = \hat{k} \times \vec{F} \tag{6.62}
$$

If $\vec{F} = \vec{r} = (ix + jy + kz)$, then

$$\begin{aligned}
[L_z, F]_x = [L_z, x] &= \hat{k} \times (ix) \\
[L_z, x] &= y \\
\text{Similarly,} \quad [\mathrm{L_z, F}]_\mathrm{y} = [\mathrm{L_z, y}] &= -x \\
[L_z, F]_z = [L_z, z] &= 0
\end{aligned} \tag{6.63}$$

Then we can write general relation as

$$[L_i, q_j] = \epsilon_{ijk} q_k$$

where $\epsilon_{ijk} = 0$ for $i = j$ or $j = k$, $\epsilon_{ijk} = 1$ for ijk are distinct and in cyclic order and $\epsilon_{ijk} = -1$ for ijk are distinct and not in cyclic order.
If $\vec{F} = \vec{P} = (ip_x + jp_y + kp_z)$, then

$$\begin{aligned}
[L_z, F]_x = [L_z, p_x] &= \hat{k} \times (ip_x) \\
[L_z, p_x] &= p_y \\
Similarly, \quad [L_z, p_y]_y = [L_z, p_y] &= -p_x \\
[L_z, F]_z = [L_z, p_z] &= 0
\end{aligned}$$

The general relation is

$$[L_i, p_j] = \epsilon_{ijk} p_k$$

If $\vec{F} = \vec{L} = (iL_x + jL_y + kL_z)$, then

$$\begin{aligned}
[L_z, F]_x = [L_z, L_x] &= \hat{k} \times (iL_x) \\
[L_z, L_x] &= L_y \\
Similarly, \quad [L_z, F]_y = [L_z, L_y] &= -L_x \\
[L_z, F]_z = [L_z, L_z] &= 0
\end{aligned} \tag{6.64}$$

Again consider

$$\begin{aligned}
[L_x, L_y] &= [yp_z - zp_y, L_y] \\
&= [yp_z, L_y] - [zp_y, L_y] \\
&= [yp_z, zp_x - xp_z] - [zp_y, zp_x - xp_z] \\
&= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_x, xp_z] \\
&= [yp_z, zp_x] + [zp_y, xp_z] \quad ([yp_z, xp_z] = 0, \quad [zp_x, zp_x] = 0) \\
&= xp_y - yp_x \\
[L_x, L_y] &= L_z, \qquad [L_y, L_x] = -L_z
\end{aligned}$$

Similarly,

$$\begin{aligned}
[L_y, L_z] &= L_x, \qquad [L_z, L_y] = -L_x \\
[L_z, L_x] &= L_y, \qquad [L_x, L_z] = -L_y
\end{aligned}$$

Thus the general relation can be written as

$$[L_i, L_j] = \epsilon_{ijk} L_k$$

where $\epsilon_{ijk} = 0$ for $i = j$ or $j = k$, $\epsilon_{ijk} = 1$ for ijk are distinct and in cyclic order and $\epsilon_{ijk} = -1$ for ijk are distinct and not in cyclic order.

6.6 Hamilton-Jacobi equation

The Hamilton-Jacobi equation makes use of a special canonical transformation to convert the standard Hamiltonian problem of $2N$ first-order ordinary differential equations in $2N$ variables into a single first-order partial differential equation with $N+1$ partial derivatives with respect to the q_i and time.
If a canonical transformation from some arbitrary set of generalized coordinates (q,p) to some new set (Q,P) such that all the Q and P are constant in time, then

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} = 0 \qquad \dot{P}_i = -\frac{\delta H}{\delta Q_i} = 0 \tag{6.65}$$

One way to guarantee the above conditions is to require that

$$H(Q,P) = 0$$

Equation for the transformation of the Hamiltonian under a canonical transformation, (equation 6.29), becomes

$$H(q,p,t) + \frac{\delta F}{\delta t} = 0 \tag{6.66}$$

Since the new momenta will be constant, it is sensible to make F a function of the type F_2, $F = S(q,P)$. Then

$$p_i = \frac{\delta S}{\delta q_i} \tag{6.67}$$

The equation 6.66 becomes,

$$H\left(q, \frac{\delta S}{\delta q_i}, t\right) + \frac{\delta S}{\delta t} = 0 \tag{6.68}$$

Equation 6.68 is called Hamilton-Jacobi equation, constitutes a partial differential equation of N independent cocordinates $q_1, q_2, q_3, ..., q_N$ and t. That is there are $N+1$ variables (N initial values of q and a constant energy). S is known as Hamilton's principle function.
Since a solution S of the equation 6.68 will generate a transformation that makes the N components of P constant, and since S is a function of the P, the P can be taken to be the N constants. Independent of the above equation, we know that there must be N additional constants to specify the full motion. These are the Q. The existence of these extra constants is not implied by the Hamilton-Jacobi equation, since it only needs $N+1$ constants to find a full solution S. The additional N constants exist because of Hamilton's equations, which require $2N$ initial conditions for a full solution.
Since the P and Q are constants, it is conventional to refer to them with the symbols $\alpha_i = P_i$ and $\beta_i = Q_i$. The full solution $q(t), p(t)$ to the problem is found by making use of the generating function S and the initial conditions $q(0)$ and $p(0)$. Then the function S is

$$S = S(q,P,t) = S(q_1, q_2, ..., q_N, \alpha_1, \alpha_2, \alpha_3, ...\alpha_{N+1}, t) = S(q, \alpha, t)$$

where quantities $\alpha_1, \alpha_2, \alpha_3, ...\alpha_{N+1}$ are $N+1$ independent constants of integration. The generating function partial derivative relations are

$$p_i = \frac{\delta S}{\delta q_i} \qquad \beta_i = \frac{\delta S}{\delta \alpha_i} = Q_i \tag{6.69}$$

The constants α and β are found by applying the above equations at $t = 0$.
The time derivative of S can be written as

$$\frac{dS}{dt} = \sum_i \frac{\delta S}{\delta q_i}\frac{dq_i}{dt} + \frac{\delta S}{\delta t} \tag{6.70}$$

By using equations 6.67 and 6.68 in equation 6.70

$$\frac{dS}{dt} = \sum_i p_i \dot{q}_i - H = L \tag{6.71}$$

$$S = \int L\ dt + constant \tag{6.72}$$

This is an interesting result - that the action integral is the generator of the canonical transformation that corresponds to time evolution.

When the Hamiltonian does not depend explicitly upon the time, Hamilton's principle function S can be written in the form

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t \tag{6.73}$$

$$\frac{\delta S}{\delta q_i} = \frac{\delta W}{\delta q_i}, \qquad \frac{\delta S}{\delta t} = -\alpha = -E \tag{6.74}$$

where $\alpha = E$ is the time independent value of H and $W(q,\alpha)$ is called *Hamilton's characteristic function.* The time derivative of $W(q,\alpha)$ is

$$\frac{dW}{dt} = \frac{\delta W}{\delta q_i}\dot{q}_i \tag{6.75}$$

By using equations 6.69 and 6.74 in equation 6.75,

$$\frac{dW}{dt} = p_i\dot{q}_i \tag{6.76}$$

$$W = \int p_i\dot{q}_i dt = \int p_i dq_i \tag{6.77}$$

which is known as the *abbreviated action.*

6.6.1 Simple harmonic oscillator

The simple harmonic oscillator Hamiltonian is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E \tag{6.78}$$

where E is the time independent value of H. The Hamilton-Jacobi equation for this Hamiltonian can be written by using equation 6.69 as

$$\frac{1}{2m}\left[\left(\frac{\delta S}{\delta q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\delta S}{\delta t} = 0 \tag{6.79}$$

Since H is conserved, $\frac{\delta S}{\delta t} = constant = -\alpha$ and by using equation 6.74

$$\frac{1}{2m}\left[\left(\frac{\delta W}{\delta q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

$$\frac{\delta W}{\delta q} = \sqrt{2m\alpha - m^2\omega^2 q^2} \tag{6.80}$$

$$W = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\; dq \tag{6.81}$$

Since $S = W - \alpha t$, then

$$S = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\; dq - \alpha t \tag{6.82}$$

$$\frac{\delta S}{\delta \alpha} = \sqrt{\frac{m}{2\alpha}} \int \frac{dq}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} - t$$

$$\beta = \frac{1}{\omega}\int \frac{\sqrt{\frac{m\omega^2}{2\alpha}}}{\sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}} dq - t$$

$$\beta + t = \frac{1}{\omega} arc\sin\sqrt{\frac{m\omega^2}{2\alpha}}q$$

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \omega\beta)$$

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \phi) \tag{6.83}$$

Again consider equation 6.80,

$$\frac{\delta W}{\delta q} = \sqrt{2m\alpha\left(1 - \frac{m\omega^2 q^2}{2\alpha}\right)}$$

$$p = \sqrt{2m\alpha[1 - \sin^2(\omega t + \phi)]} = \sqrt{2m\alpha}\,\cos(\omega t + \phi) \tag{6.84}$$

At time $t = 0$, the equations 6.83 and 6.84 becomes

$$q_o = \sqrt{\frac{2\alpha}{m\omega^2}} \sin\phi \quad (6.85)$$

$$p_o = \sqrt{2m\alpha} \cos\phi \quad (6.86)$$

On squaring and adding the above equations, we get

$$2m\alpha = p_o^2 + m^2\omega^2 q_o^2 \quad (6.87)$$

Thus α can be obtained in terms of p_o and q_o. Equations 6.85/6.86 gives

$$\tan\phi = m\omega\frac{q_o}{p_o} \quad (6.88)$$

When $q_o = 0$, $\beta = 0$ corresponds to starting the motion with the oscillator at its equilibrium position $q = 0$. Thus Hamilton's principle function is the generator of a canonical transformation to a new coordinate that measures the phase angle of the oscillation and to a new canonical momentum identified as the total energy. By using equation 6.81, the Hamilton's principle function can be written as

$$S = \sqrt{2m\alpha} \int \sqrt{1 - \frac{m\omega^2 q^2}{2\alpha}}\, dq - \alpha t$$

On substituting for q and dq by using equation 6.83,

$$\begin{aligned} S &= 2\alpha \int \cos^2(\omega t + \phi) dt - \alpha t \\ &= \alpha \int \left(2\cos^2(\omega t + \phi) - 1\right) dt \\ S &= \alpha \int \left(\cos^2(\omega t + \phi) - \sin^2(\omega t + \phi)\right) dt \quad (6.89) \end{aligned}$$

from equations 6.83 and 6.84, we can get

$$S = \int \left(\frac{p^2}{2m} - \frac{m\omega^2 q}{2}\right) dt \quad (6.90)$$

$$S = \int L\, dt \quad (6.91)$$

i.e., S is the time integral of the Lagrangian.