Nuclear Shell Model

It has been found that the nuclei with proton number or neutron number equal to certain numbers 2,8,20,28,50,82 and 126 behave in a different manner when compared to other nuclei having neighboring values of Z or N. Hence these numbers are known as magic numbers. This is found to be in accordance with the observed nature of elements with filled shells. Thus Physicists looked at such a possibility in case of filling of nucleons in the nucleus. Thus a new model of nucleus has emerged. This model is known as the Shell model.

Experimental evidences for the existence of magic numbers;

1. The binding energy of magic numbered nuclei is much larger than the neighboring nuclei. Thus larger energy is required to separate a single nucleon from such nuclei.
2. Number of stable nuclei with a given value of Z and N corresponding to the magic number are much larger than the number of stable nuclei with neighboring values of Z and N. For example, Sn with Z=50 has 10 stable isotopes, Ca with Z=20 has six stable isotopes.
3. Naturally occurring isotopes whose nuclei contain magic numbered Z or N have greater relative abundances. For example, Sr-88 with N=50, Ba-138 with N=82 and Ce-140 with N=82 have relative abundances of 82.56%, 71.66% and 88.48% respectively.
4. Three naturally occurring radioactive series decay to the stable end product Pb with Z=82 in three isotopic forms having N=126 for one of them.
5. Neutron absorbing cross section is very low for the nuclei having magic numbered neutron number.
6. Nuclei with the value of N just one more than the magic number spontaneously emit a neutron (when excited by preceding beta-decay) E.g., O-17, K-87 and Xe-137.
7. Nuclei with magic numbers of neutrons or protons have their first excited states at higher energies than in cases of the neighboring nuclei.
8. Electric quadrupole moment of magic numbered nuclei is zero indicating the spherical symmetry of nucleus for closed shells.
9. Energy of alpha or beta particles emitted by magic numbered radioactive nuclei is larger than that from other nuclei.

Independent particle model

Consider a nucleon moving independently in the harmonic oscillator potential which is spherically symmetric. The Schrodinger equation given below can be solved in the Cartesian coordinate system as well as in the spherical coordinate system.

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r})\right)\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}),$$

with -------(1)

$$V(\boldsymbol{r}) = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2).$$

In the Cartesian coordinate system, it can be separated into three linear harmonic oscillators and consequently, the energy of the three-dimensional harmonic oscillator is the sum of the energies of the three linear harmonic oscillators.

$$E = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right) = \hbar\omega\left(N + \frac{3}{2}\right),$$ -----(2)

where N is the principal quantum number which will assume positive integral values including zero. From Eq. (2), we observe that. the energy depends only on the quantum number N and not on n_x, n_y and n_z. So, there is a degeneracy of energy levels. Since the proton (or neutron) has spin $1/2\hbar$, there are two possible spin orientations for each state specified by a set of quantum numbers n_x, n_y and n_z. Applying the Pauli exclusion principle, we obtain the number of protons or neutrons that can have a particular energy as shown in Table I.

Table I: Number (N_N) of protons or neutrons in each state with the principal quantum Number N. The multiplicative factor 2 in the last column is due to two spin states.

N	n_x	n_y	n_z	$\mathcal{N}_N$
0	0	0	0	$1\times 2=2$
1	1	0	0	
	0	1	0	
	0	0	1	$3\times 2=6$
2	1	1	0	
	1	0	1	
	0	1	1	
	2	0	0	
	0	2	0	
	0	0	2	$6\times 2=12$

In general

$$\mathcal{N}_N=(N+1)(N+2).$$ ----(3)

However, it is found not convenient to work in Cartesian coordinate system. Since the potential is spherically symmetric, one can attempt to solve the Schrodinger equation in spherical coordinates.

Let us write ∇^2 in spherical coordinates.

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{r^2\hbar^2},\end{aligned}$$ -------(4)

where L^2 is the square of the angular momentum operator

$$L^2=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right],$$ ----------------------------------(5)

which has the spherical harmonics $Y_{lm}(\theta,\phi)$ as eigen functions with eigen values $l(l+1)\hbar^2$.

$$L^2Y_{lm}(\theta,\phi)=l(l+1)\hbar^2Y_{lm}(\theta,\phi).$$ -------------------------------(6)

Substituting (4) in (1) we get,

$$\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{L^2}{r^2\hbar^2}+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}m\omega^2r^2\right)\right\}\psi(\boldsymbol{r})=0.$$ --------------(7)

Writing the solution of ψ(r) of eq. (7) as a product of radial and angular functions

$$\psi(\mathbf{r}) = R(r)\, Y_{lm}(\theta, \phi),$$ -------------------------------(8)

And using the eigen value equation (6), we obtain the radial equation

$$\left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) - \frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}m\omega^2 r^2\right)\right\} R(r) = 0.$$ -----------(9)

Applying the bound state boundary condition that the radial function should vanish at infinity, we obtain the discrete energy levels of the three dimensional harmonic oscillator.

$$E = \left(2n + l + \frac{3}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, \cdots.$$ ------------------------------------(10)

Comparing (10) with the energy levels (2) obtained in the Cartesian coordinate system,

$$E_N = \left(N + \frac{3}{2}\right)\hbar\omega = \left(2n + l + \frac{3}{2}\right)\hbar\omega,$$ ---------------------(11)

One can find

$$N = 2n + l, \qquad n = 0, 1, 2, \cdots.$$ ---------------------(12)

The eigenvalues and eigenfunctions in spherical coordinate system depend on two quantum numbers: the radial quantum number *n* and the orbital quantum number *l.* It follows that for a given total quantum number *N,* the orbital quantum number *l* can take only even values if *N* is even or only odd values if *N* is odd.

$$\begin{aligned} l &= N - 2n \\ &= N, N-2, N-4, \cdots, 0 \text{ or } 1. \end{aligned}$$

The additional quantum number m occurring in Eq. (6.8) is the magnetic quantum number. It may be remarked that the accidental degeneracy in the states with a given *l* but with different m having the same energy, occurs in any spherical potential. However, in the case of the oscillator potential, there is a special degeneracy as well, i.e., states with different *l* values but with the same *N* have the same energy. Accordingly, the number of protons or neutrons with a given value of *N* is

$$\mathcal{N}_N = \sum_l 2(2l+1),$$ ------------------(13)

where the factor 2 is due to the two spin states. Equation (6.12) gives the different *l* values allowed for a given *N.* The permitted values of *l* are all even or odd and consequently, the parity (given by *(-1)')* of all the nucleons with a given quantum number *N* is the same.

The three dimensional harmonic oscillator energy levels are labelled by a pair of quantum numbers *(n, l).* Also the spectroscopic notation *s, p, d, f, g, h, .. ,* is used to denote states with *l* = 0,1,2,3,4,5, .. '.

(n, l) : Os; Op; (Od, Is); (Of, Ip); (Og, Id, 2s);
(Oh, If, 2p); (Oi, *Ig, 2d, 3s); (OJ, Ih, 2f, 3p).* ---------------(14)

The bracketed levels on the right hand side are degenerate. We also give below the normalized radial functions for a few *(n, l)* values.

$$R_{nl}(r) = N_{nl}\,\alpha^{3/2} e^{-\alpha^2 r^2/2} (\alpha r)^l f(r^2),$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}}$$

and

$$\begin{aligned} f(r^2) &= 1, && \text{for } n = 0; \\ &= \frac{2l+3}{2} - \alpha^2 r^2, && \text{for } n = 1; \\ &= \frac{1}{\sqrt{2}}\left\{\frac{(2l+3)(2l+5)}{4} - (2l+5)(\alpha r)^2 + (\alpha r)^4\right\}, && \text{for } n = 2. \end{aligned}$$

The normalization factor N_{nl} is given by

$$N_{nl} = \left\{\frac{2^{n+l+2}}{\sqrt{\pi}(2n+2l+1)!!}\right\}^{1/2},$$

such that

$$\int_0^\infty |R_{nl}(r)|^2\, r^2 dr = 1.$$

In literature on nuclear shell model[2], a slightly different radial quantum number n_r (to distinguish it from the radial quantum number n, we use the notation n_r) is also used such that

$$N = 2n_r + l - 2 \qquad \text{and} \qquad n_r = n + 1, \quad n_r = 1, 2, 3, \cdots.$$

In general usage, the oscillator energy levels are labelled by a pair of quantum numbers (n, l) as described in (6.14) or (n_r, l) as shown below.

$$\begin{aligned} (n_r, l) \quad : \quad & 1s; 1p; (1d, 2s); (1f, 2p); (1g, 2d, 3s); \\ & (ih, 2f, 3p); (1i, 2g, 3d, 4s); (1j, 2h, 3f, 4p). \end{aligned}$$

The radial quantum number *nr* denotes that it is the nth time that the *l* value occurs in the scheme. Alternatively, *nr* can be interpreted as the number of nodes that occurs in the radial wave function including the one at infinity. The bracketed levels on the right hand side of (6.20) are degenerate. We shall use the notation *(n, I)* or *(n_r, I)* as the situation demands.

In Table II, we present the number of protons or neutrons that can occupy each oscillator state with total quantum number *N.*

Table II: Number of protons or neutrons occupying each state with total quantum number *N* and the orbital substates with quantum numbers *n_r, l.* The energy *E* is in units of ħω.

N	E	Orbitals (n_r, l)	$\mathcal{N}_N = (N+1)(N+2) = \sum_l 2(2l+1)$	$\sum_N \mathcal{N}_N$
0	3/2	$1s$	2	2
1	5/2	$1p$	6	8
2	7/2	$1d, 2s$	12	20
3	9/2	$1f, 2p$	20	40
4	11/2	$1g, 2d, 3s$	30	70
5	13/2	$1h, 2f, 3p$	42	112
6	15/2	$1i, 2g, 3d, 4s$	56	168

The numbers in the last column should correspond to the closed shell nuclei and hence to the magic numbers but the discrepancy arises after the first three shells. Further, there is a degeneracy of states with different *l* values. Such a degeneracy does not exist in the case of square well potential. Empirically, let us assume a level split of different *l* orbitals with the energy given by

$$E_{N,l} = \hbar\omega\left(N + \frac{3}{2}\right) + Dl(l+1),$$

assuming an additional force which lowers the states of larger *I*. A value *D = -0.0225/* ħω is found to be satisfactory.

However, the introduction of this additional force only removes the degeneracy with respect to *l* but will not yield the correct magic numbers above 20. So, Mayer and Jenson independently in 1949 introduced a spin-orbit coupling *l* . 8. *Originally, the spin-orbit force was introduced phenomenologically to reproduce the magic numbers but now it is believed to arise from the tensor force of the nucleon-nucleon interaction.*

$$\begin{aligned}
\boldsymbol{l}\cdot\boldsymbol{s} &= \frac{1}{2}\left(\boldsymbol{j}^2 - \boldsymbol{l}^2 - \boldsymbol{s}^2\right) \\
\langle \boldsymbol{l}\cdot\boldsymbol{s}\rangle &= \frac{1}{2}\left\{j(j+1) - l(l+1) - \tfrac{1}{2}(\tfrac{1}{2}+1)\right\} \\
&= \begin{cases} \frac{1}{2}l & \text{for } j = l + \frac{1}{2}, \\ -\frac{1}{2}(l+1) & \text{for } j = l - \frac{1}{2}. \end{cases}
\end{aligned}$$

Thus we see that the spin–orbit force removes the degeneracy of the level with respect to spin and from (6.22), we find the spacing between the energy levels with $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ to be proportional to $(2l+1)V_{SO}$. Let us assume the spin–orbital potential V_{SO} to be of the Thomas form,

$$V_{SO} = U_{SO}\, \boldsymbol{l} \cdot \boldsymbol{s} = -\frac{1}{r}\frac{\partial V}{\partial r}\, \boldsymbol{l} \cdot \boldsymbol{s}.$$

For the harmonic oscillator potential, $V \sim r^2$ and consequently $(1/r)(\partial V/\partial r)$ is a constant. The negative sign in expression (6.23) lowers the levels with higher j value. By this artifice, we are able to produce bunching of levels and a large spacing between any two bunches. The magic number corresponds to the number of protons or neutrons that occupy the bunch of closely spaced single particle levels separated by large spacing. From Table 6.5 as well as from Fig. 6.1, we find that the magic numbers correspond to fairly large separations of closely spaced single particle states. It is found empirically that the correct level scheme can be obtained from the expression

$$E = \left(N + \frac{3}{2}\right)\hbar\omega + Dl(l+1) + C\, \boldsymbol{l} \cdot \boldsymbol{s},$$

with $D = -0.0225\,\hbar\omega$ and $C = -0.1\,\hbar\omega$. We now obtain the magic numbers correctly. Even more important is that we get the correct assignment for the spin and parity of the ground state of the odd-mass nuclei, assuming that the spin of the odd-mass nucleus is equal to the spin of the last nucleon.

Table 6.5: The inclusion of spin-orbit coupling causes the grouping of levels into various shells and explains the magic numbers of protons or neutrons that correspond to the closed shell structures.

Shell	States	No. of protons or neutrons in the shell	Total number of protons or neutrons
I	$1s$	2	2
II	$1p_{3/2}$, $1p_{1/2}$	4, 2	8
III	$1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$	6, 2, 4	20
IV	$1f_{7/2}$	8	28
V	$2p_{3/2}$, $1f_{5/2}$, $2p_{1/2}$, $1g_{9/2}$	4, 6, 2, 10	50
VI	$1g_{7/2}$, $2d_{5/2}$ $2d_{3/2}$, $3s_{1/2}$, $1h_{11/2}$	8, 6, 4, 2, 12	82

The single-particle energy level scheme with spin-orbit coupling is shown in Fig. I for the nuclear shell model yielding correctly the magic numbers.

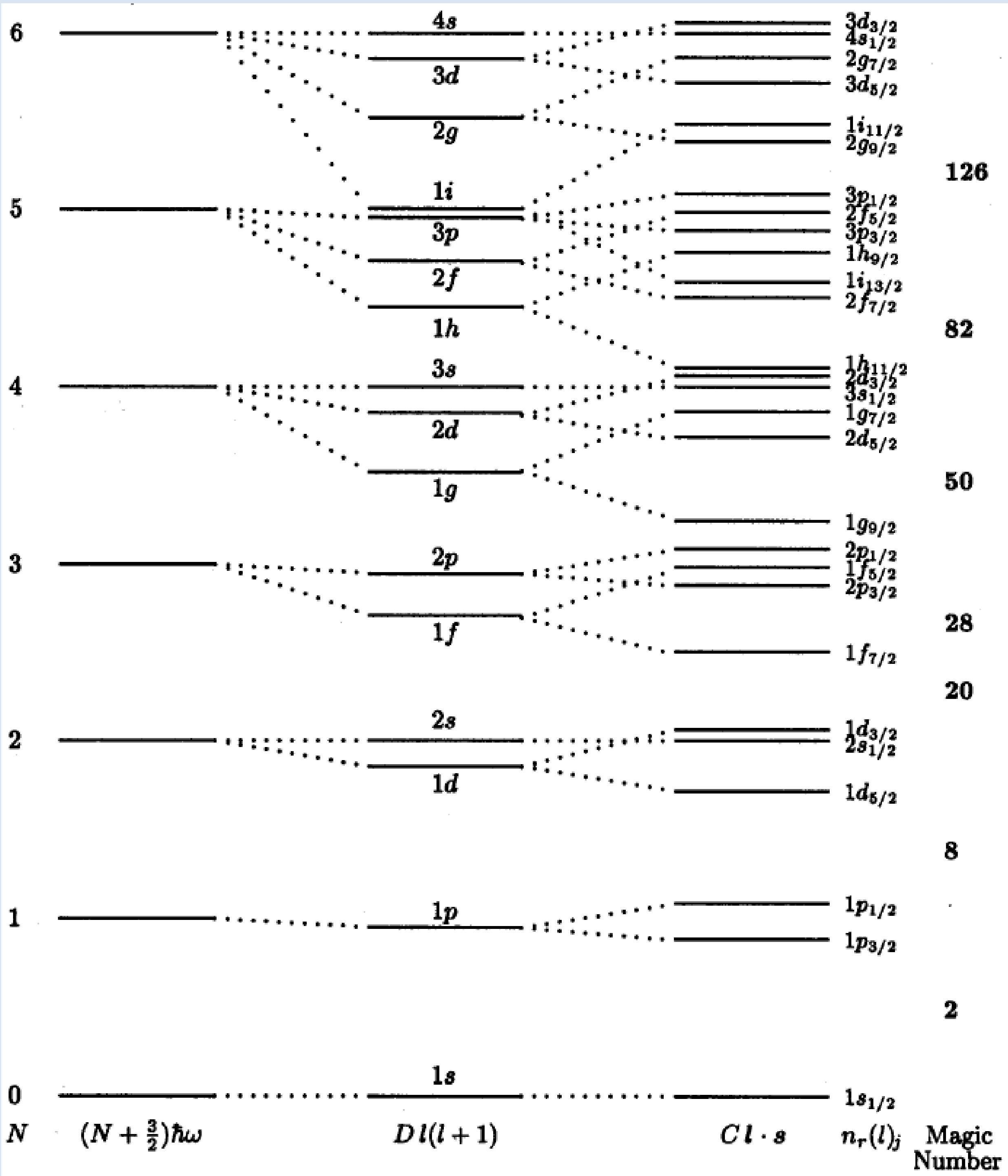

Fig 1:

Single-particle energy level scheme for nuclear shell model.