Continuum Mechanics

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1. Strain tensor

An elastic body under an applied load deforms into a new shape. As shown in the figure 1 two adducent points p and q on the body displaced to p' and q' by the application of the force $\vec{F}$. The displacements $\vec{u}_1$ and $\vec{u}_2$ of the point p and q are

$$\begin{aligned} \vec{u}_1 &= \vec{r_1}' - \vec{r}_1 \\ \vec{u}_2 &= \vec{r_2}' - \vec{r}_2 \\ \vec{u}_2 - \vec{u}_1 &= (\vec{r_2}' - \vec{r_1}') - (\vec{r}_2 - \vec{r}_1) \\ d\vec{u} &= d\vec{r}' - d\vec{r} \\ d\vec{r}' &= d\vec{u} + d\vec{r} \qquad (1) \end{aligned}$$

In terms of the cartesian components, the equation 1 can be written as

$$\sqrt{{dx'_1}^2 + {dx'_2}^2 + {dx'_3}^2} = \sqrt{(du_1 + dx_1)^2 + (du_2 + dx_2)^2 + (du_3 + dx_3)^2}$$

Using general summation rule, we can write

$$\sum_i dx_i'^2 = \sum_i (du_i + dx_i)^2$$
$$\sum_i dx_i'^2 = \sum_i dx_i^2 + 2du_i dx_i + du_i^2 \qquad (2)$$

Since $u_i <<$, the third term is quadratic in u_i, can be neglected. Then the equation 2 becomes

$$\sum_i dx_i'^2 = \sum_i dx_i^2 + 2du_i dx_i \qquad (3)$$

The displacement vector u_i is

$$u_i = u_i(x_1, x_2, x_3)$$
$$du_i = \frac{\delta u_i}{\delta x_j} dx_j \qquad (4)$$

Equation 4 in 3,

$$\sum_i dx_i'^2 = \sum_i dx_i^2 + 2\sum_i \sum_j \frac{\delta u_i}{\delta x_j} dx_j dx_i$$
$$\sum_i dx_i'^2 = \sum_i dx_i^2 + 2\sum_i \sum_j a_{ij} dx_j dx_i \qquad (5)$$

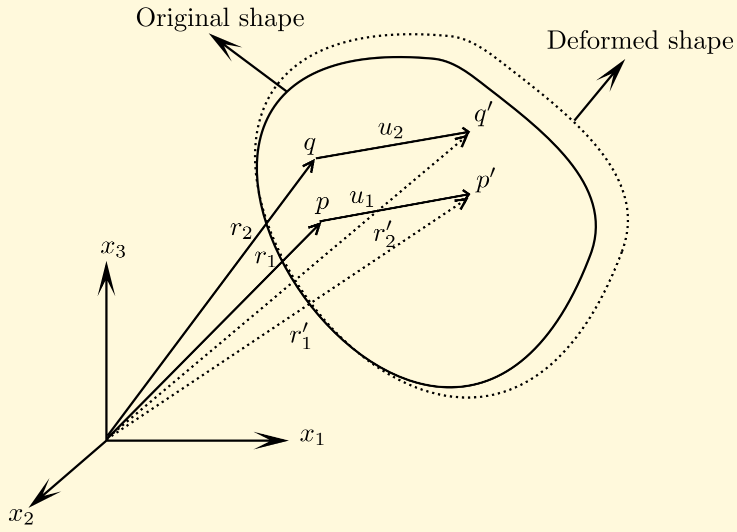

Figure 1: Deformation of an elastic body

where the tensor

$$a_{ij} = \frac{\delta u_j}{\delta x_i}$$

is the strain produced in the ith components of the deformation along jth axis and it can be written as

$$\begin{aligned} a_{ij} &= \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) \\ &= \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i}\right) + \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} - \frac{\delta u_j}{\delta x_i}\right) \\ &= \varepsilon_{ij} + \omega_{ij} \end{aligned} \tag{6}$$

where

$$\omega_{ij} = \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} - \frac{\delta u_j}{\delta x_i}\right)$$

is the antisymmetric part of the tensor a_{ij}, and it can be shown that it corresponds to a pure rotation of the body as a whole.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i}\right) \tag{7}$$

is the symmetric part of the tensor a_{ij}, and is called the *strain tensor*. In cartesian components,

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\delta u_x}{\delta x} & \frac{1}{2}\left(\frac{\delta u_x}{\delta y} + \frac{\delta u_y}{\delta x}\right) & \frac{1}{2}\left(\frac{\delta u_x}{\delta z} + \frac{\delta u_z}{\delta x}\right) \\ \frac{1}{2}\left(\frac{\delta u_y}{\delta x} + \frac{\delta u_x}{\delta y}\right) & \frac{\delta u_y}{\delta y} & \frac{1}{2}\left(\frac{\delta u_y}{\delta z} + \frac{\delta u_z}{\delta y}\right) \\ \frac{1}{2}\left(\frac{\delta u_z}{\delta x} + \frac{\delta u_x}{\delta z}\right) & \frac{1}{2}\left(\frac{\delta u_z}{\delta y} + \frac{\delta u_y}{\delta z}\right) & \frac{\delta u_z}{\delta z} \end{bmatrix} \quad (8)$$

we can that the strain tensor ε_{ij} is symmetric. The eigenvectors of ε_{ij} are the principal directions of the strain, i.e., the directions where there is no shear strain. The eigenvalues, $\varepsilon_{11} = \lambda_1, \varepsilon_{22} = \lambda_2$ and $\varepsilon_{33} = \lambda_3$ are the principal strains and give the unit elongations in the principal directions.

When there is no rotation, $\omega_{ij} = 0$, and $a_{ij} = \varepsilon_{ij}$. The equation 5 becomes,

$$\sum_i dx_i'^2 = \sum_i dx_i^2 + 2\sum_i \sum_j \varepsilon_{ij} dx_j dx_i \quad (9)$$

If the strain tensor is diagonalize at a given point, the equation 9 can be written as,

$$\sum_i dx_i'^2 = \sum_{ij} \delta_{ij} dx_i dx_j + 2 \sum_{ij} \varepsilon_{ij} dx_i dx_j$$

$$= \sum_{ij} (\delta_{ij} + 2\varepsilon_{ij})\, dx_i dx_j$$

$$dx_i'^2 = (1 + 2\varepsilon_{ii})\, dx_i^2$$

$$dx_i' = \sqrt{1 + 2\varepsilon_{ii}}\, dx_i \quad (10)$$

By using Binomial expansion and by neglecting the higher terms, the equation 10 can be written as

$$dx_i' = (1 + \varepsilon_{ii})\, dx_i \quad (11)$$

Then the deformed volume element can be written as

$$dx'\, dy'\, dz' = (1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33})\, dx\, dy\, dz \quad (12)$$

On simplifying the equation 12 and by neglecting the higher terms,

$$
\begin{aligned}
dV' &= (1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\, dV \\
&= \left(1 + \sum_i \varepsilon_{ii}\right) dV \qquad (13) \\
\sum_i \varepsilon_{ii} &= \frac{dV' - dV}{dV} \qquad (14)
\end{aligned}
$$

i.e., the sum of the diagonal components of the strain tensor becomes the relative volume change.

2. Stress tensor

The internal forces which occur when a body is deformed are called internal stresses. If no deformation occurs, there are no internal stresses. The internal stresses are due to molecular forces, i.e. the forces of interaction between the molecules. The molecular forces have a very short range of action.

If $\vec{F}$ is the force acting per unit volume on the body, the force acting on the volume element dV is $\sum_i \vec{F}_i \, dV$. Then the net force acting on the body is

$$\vec{F}_T = \iiint \sum_i \vec{F}_i \, dV \tag{15}$$

The total force can be obtained from an integral of a vector $\vec{F}_i$, then the vector F_i must be the divergence of a tensor of rank two, i.e. be of the form

$$\vec{F}_i = \frac{\delta \sigma_{ij}}{\delta x_j} \tag{16}$$

Equation 16 in equation 15,

$$\vec{F}_T = \iiint \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} \, dV \tag{17}$$

According to the Green's theorem,

$$\iiint \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} \, dV = \iint \sum_{ij} \sigma_{ij} \, ds_j \tag{18}$$

where ds_j are the components of the surface element vector ds, directed along the outward normal. The tensor σ_{ij} is called the stress tensor. $\sigma_{ij} \, ds_j$ is the ith component of the force on the surface element ds_j. The stress vector acting on

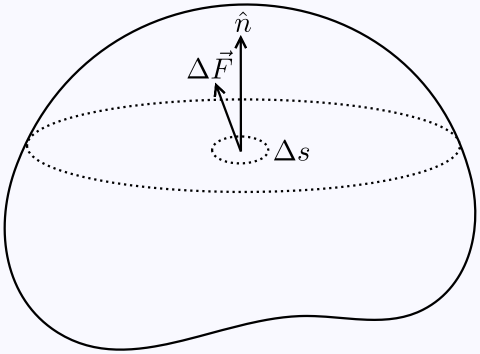

Figure 2: Definition of a stress vector

any plane with normal **n** can be defined as

$$
\begin{aligned}
t_n &= \lim_{\Delta s_j \to 0} \frac{\Delta \vec{F}_i}{\Delta s_j} \hat{n} \\
t_n &= \sigma_{ij} \hat{n}
\end{aligned} \tag{19}
$$

where the tensor σ_{ij} is

$$
\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{20}
$$

The diagonal elements σ_{11}, σ_{22} and σ_{33} are the normal stresses and the off-diagonal elements $\sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}$ and σ_{32} are the shear stresses.

The moment of the forces on a portion of the body can be written as an anti-symmetrical tensor of rank two, whose components are $F_i x_j - F_j x_i$, where x_i are the co-ordinates of the point where the force is applied. Hence the moment of the forces on the volume element dV is

$$
M_{ij} = \iiint (F_i x_j - F_j x_i)\, dV \tag{21}
$$

Substituting the expression 16 for F_i, we find

$$\begin{aligned} M_{ij} &= \iiint \left(\frac{\delta \sigma_{ik}}{\delta x_k} x_j - \frac{\delta \sigma_{jk}}{\delta x_k} x_i \right) dV \\ &= \iiint \frac{\delta}{\delta x_k} \left(\sigma_{ik} x_j - \sigma_{jk} x_i \right) dV - \iiint \left(\sigma_{ik} \frac{\delta x_j}{\delta x_k} - \sigma_{jk} \frac{\delta x_i}{\delta x_k} \right) dV \end{aligned} \tag{22}$$

By applying Green's theorem to the equation 22,

$$= \iint \left(\sigma_{ik} x_j - \sigma_{jk} x_i \right) ds_k - \iiint \left(\sigma_{ik} \frac{\delta x_j}{\delta x_k} - \sigma_{jk} \frac{\delta x_i}{\delta x_k} \right) dV \tag{23}$$

The derivative of a co-ordinate with respect to itself is unity, and with respect to another co-ordinate is zero. Thus $\frac{\delta x_j}{\delta x_k} = \delta_{jk}$, where δ_{jk} is the unit tensor. The equation 22 becomes

$$\begin{aligned} M_{ij} &= \iint \left(\sigma_{ik} x_j - \sigma_{jk} x_i \right) ds_k - \iiint \left(\sigma_{ik} \delta_{jk} - \sigma_{jk} \delta_{ik} \right) dV \\ &= \iint \left(\sigma_{ik} x_j - \sigma_{jk} x_i \right) ds_k - \iiint \left(\sigma_{ij} - \sigma_{ji} \right) dV \end{aligned} \tag{24}$$

Since M_{ij} is an integral over the surface only, the second term must vanish. Then we must have

$$\sigma_{ij} = \sigma_{ji}$$

That is, the stress tensor is symmetrical. The moment of the forces on a portion of the body can then be written as

$$M_{ij} = \iint (\sigma_{ik} x_j - \sigma_{jk} x_i)\, ds_k \tag{25}$$

In equilibrium the internal stresses in every volume element must balance, i.e. we must have $\sum_i F_i = 0$. Thus the equations of equilibrium for a deformed body are

$$\sum_i F_i = \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} = 0 \tag{26}$$

If the body is in a gravitational field, then

$$\sum_i F_i = \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} + \rho g = 0 \tag{27}$$

where ρ is the density and g is the gravitational acceleration vector, directed vertically downwards.

If P_i be the external force on unit area of the surface of the body, so that a force $P_i\, df$ acts on a surface element df. In equilibrium, this must be balanced by the force $-\sigma_{ij} df_j$ of the internal stresses acting on that element. Thus we must have

$$\begin{aligned} P_i\, df &= \sigma_{ij} df_j \\ P_i\, df &= \sigma_{ij} \hat{n}_j\, df \\ P_i &= \sigma_{ij} \hat{n}_j \end{aligned} \tag{28}$$

This is the condition which must be satisfied at every point on the surface of a body in equilibrium.

3. Thermodynamics of deformation

The work done W by the internal stresses in the deformed body is

$$W = \iiint \delta W \, dV = \iiint \sum_i F_i \, \delta u_i dV \tag{29}$$

where δW is the workdone by the internal stresses per unit volume and δu_i is the displacement vector due to F_i. On substituting for F_i in terms of stress tensor, we have

$$\begin{aligned} \iiint \delta W \, dV &= \iiint \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} \delta u_i dV \\ &= \sum_{ij} \iint \sigma_{ij} \delta u_i ds_j - \iiint \sum_{ij} \sigma_{ij} \frac{\delta}{\delta x_j} (\delta u_i) \, dV \end{aligned} \tag{30}$$

By considering an infinite medium which is not deformed at infinity, we make the surface of integration in the first integral tend to infinity; then $\sigma_{ij} = 0$ on the surface, and the integral is zero. Then equation 30 becomes

$$\begin{aligned}
\iiint \delta W \, dV &= -\iiint \sum_{ij} \sigma_{ij} \, \delta \left(\frac{\delta u_i}{\delta x_j} \right) dV \\
&= -\iiint \sum_{ij} \sigma_{ij} \, \delta \left[\frac{1}{2} \left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i} \right) + \frac{1}{2} \left(\frac{\delta u_i}{\delta x_j} - \frac{\delta u_j}{\delta x_i} \right) \right] dV \\
&= -\iiint \sum_{ij} \sigma_{ij} \delta(\varepsilon_{ij} + \omega_{ij}) dV \qquad (31)
\end{aligned}$$

where

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i} \right)$$

is the *strain tensor* and

$$\omega_{ij} = \frac{1}{2} \left(\frac{\delta u_i}{\delta x_j} - \frac{\delta u_j}{\delta x_i} \right)$$

is the tensor corresponds to a pure rotation of the body. Since there is no rotation, $\omega_{ij} = 0$, then the equation 31 becomes

$$\begin{aligned} \iiint \delta W \, dV &= -\iiint \sum_{ij} \sigma_{ij} \delta\varepsilon_{ij} dV \\ \delta W &= -\sum_{ij} \sigma_{ij} \delta\varepsilon_{ij} \\ dW &= -\sum_{ij} \sigma_{ij} d\varepsilon_{ij} \end{aligned} \tag{32}$$

Equation 32 gives the workdone by the internal stresses per unit volume.

According to I law of thermodynamics,

$$\begin{aligned} dQ &= dU + dW \\ dU &= dQ - dW \\ dU &= T \, dS + \sum_{ij} \sigma_{ij} \, \delta\varepsilon_{ij} \qquad \text{(from equation 32)} \end{aligned} \tag{33}$$

The equation 33 gives the thermodynamic identity for deformed bodies. The Helmholtz potential or free energy F of the body is

$$\begin{aligned} F &= U - TS \\ dF &= dU - T\,dS - S\,dT \end{aligned} \tag{34}$$

Equation 33 in equation 34 gives

$$dF = \sum_{ij} \sigma_{ij}\delta\varepsilon_{ij} - S\,dT \tag{35}$$

The thermodynamic potential Φ is given by

$$\begin{aligned} \Phi &= F + PV \\ \Phi &= F - \sum_{ij} \sigma_{ij}\varepsilon_{ij} \\ d\Phi &= dF - \sum_{ij} \sigma_{ij}\,d\varepsilon_{ij} - \sum_{ij} \varepsilon_{ij}\,d\sigma_{ij} \end{aligned} \tag{36}$$

Equation 35 in equation 36 gives,

$$d\Phi = -S\,dT - \sum_{ij} \varepsilon_{ij}\,d\sigma_{ij} \tag{37}$$

For constant entropy or constant temperature, the equations 33 and equations 35 gives

$$\sigma_{ij} = \left(\frac{\delta U}{\delta \varepsilon_{ij}}\right)_S = \left(\frac{\delta F}{\delta \varepsilon_{ij}}\right)_T \tag{38}$$

For constant temperature, the equations 37 becomes

$$\varepsilon_{ij} = -\left(\frac{\delta \Phi}{\delta \sigma_{ij}}\right)_T \tag{39}$$

3.1. Hooke's law

Consider an isotropic body in small deformation at constant temperature throughout the body. Then the stress tensor is given by

$$\sigma_{ij} = \left(\frac{\delta F}{\delta \varepsilon_{ij}}\right)_T \tag{40}$$

Thus free energy F be an explicit function of T and ε_{ij}. At constant temperature, it can be expand in powers of ε_{ij} using Taylor's series.

$$F = F_o + \sum_{ij} C_{ij}\varepsilon_{ij} + \frac{1}{2!}\sum_{ij} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \ldots\ldots\ldots\ldots\ldots \tag{41}$$

At first consider the undeformed state of the body in the absence of external forces at the same temperature. In undeformed state $\varepsilon_{ij} = 0$, and also the internal stresses $\sigma_{ij} = 0$. Since, $\sigma_{ij} = \left(\frac{\delta F}{\delta \varepsilon_{ij}}\right)_T$, it follows that there is no linear term in the expansion of F in powers of ε_{ij} and for small deformations higher order terms can be neglected. Therefore F can be written as

$$F \;=\; F_o + \frac{1}{2!}\sum_{ij} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \ldots\ldots\ldots\ldots\ldots \tag{42}$$

where C_{ijkl} is a tensor of rank four, called the elastic modulus tensor.

Since the free energy F is a scalar, each term in the expansion of F must be a scalar also. Two independent scalars of the second degree can be formed from the components of the symmetrical tensor ε_{ij}; they can be taken as the squared sum of the diagonal components ε_{ii}^2 and the sum of the squares of all the components ε_{ij}^2. Then the equation 42 can be written as,

$$F \;=\; F_o + \frac{1}{2}\sum_{ij} \lambda\varepsilon_{ii}^2 + \sum_{ij} \mu\varepsilon_{ij}^2 \tag{43}$$

This is the general expression for the free energy of a deformed isotropic body.

λ and μ are called *Lamé coefficients*.

According to equation 14, the change in volume in the deformation is given by the sum ε_{ii}. If this sum is zero, then the volume of the body is unchanged by the deformation, only its shape being altered. Such a deformation is called a *pure shear*. The opposite case is that of a deformation which causes a change in the volume of the body but no change in its shape. Such a deformation is called a *hydrostatic compression*.

Thus any deformation can be represented as the sum of a pure shear and a hydrostatic compression. Then ε_{ij} can be expressed as

$$\varepsilon_{ij} = \left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right) + \frac{1}{3}\delta_{ij}\varepsilon_{ll} \tag{44}$$

where δ_{ij} is the Kronecker delta. The first term on the right is evidently a pure shear and the second term is a hydrostatic compression. On substituting the equation 44 in equation 43,

$$
\begin{aligned}
F &= F_o + \frac{1}{2}\sum_i \lambda\varepsilon_{ii}^2 + \mu\sum_{ijl}\left[\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right) + \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right]^2 \\
&= F_o + \frac{1}{2}\sum_i \lambda\varepsilon_{ii}^2 + \mu\sum_{ijl}\left[\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)^2 + \frac{1}{9}\delta_{ij}^2\varepsilon_{ll}^2 + \frac{2}{3}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)\delta_{ij}\varepsilon_{ll}\right] \\
&= F_o + \frac{1}{2}\sum_i \lambda\varepsilon_{ii}^2 + \mu\left[\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)^2 + \frac{2}{3}\sum_{ijl}\varepsilon_{ij}\delta_{ij}\varepsilon_{ll} - \frac{1}{9}\sum_{ijl}\delta_{ij}^2\varepsilon_{ll}^2\right] \qquad (45)
\end{aligned}
$$

The last terms corresponding to hydrostatic compression, so that we can take

$i = j$, and obviously $\varepsilon_{ii} = \varepsilon_{ll}$, then the equation 45 can be written as

$$\begin{aligned} F &= F_o + \frac{1}{2}\sum_i \lambda\varepsilon_{ll}^2 + \mu\left[\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)^2 + \frac{2}{3}\sum_l \varepsilon_{ll}^2 - \frac{1}{3}\sum_l \varepsilon_{ll}^2\right] \\ &= F_o + \mu\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)^2 + \frac{1}{2}\sum_l \lambda\varepsilon_{ll}^2 + \frac{\mu}{3}\sum_l \varepsilon_{ll}^2 \\ &= F_o + \mu\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)^2 + \frac{1}{2}\left(\lambda + \frac{2}{3}\mu\right)\sum_l \varepsilon_{ll}^2 \\ F &= F_o + \mu\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)^2 + \frac{1}{2}K\sum_l \varepsilon_{ll}^2 \qquad (46) \end{aligned}$$

where $K = (\lambda + \frac{2}{3}\mu)$ is called *modulus of hydrostatic compression (or bulk* *modulus)* and μ is called *modulus of rigidity.*

$$\begin{aligned} dF &= \mu \sum_{ijl} 2\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)(d\varepsilon_{ij} - \frac{1}{3}\delta_{ij}d\varepsilon_{ll}) + K\sum_{l}\varepsilon_{ll}\, d\varepsilon_{ll} \\ &= 2\mu\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)d\varepsilon_{ij} - \frac{2}{3}\mu\sum_{ijl}\varepsilon_{ij}\delta_{ij}d\varepsilon_{ll} + \frac{2}{9}\mu\sum_{ijl}\delta_{ij}^2\varepsilon_{ll}\, d\varepsilon_{ll} \\ &\quad + K\sum_{l}\varepsilon_{ll}\, d\varepsilon_{ll} \end{aligned} \tag{47}$$

By taking $i = j$ in hydrostatic compression terms (second and third terms) and writing $d\varepsilon_{ll} = \delta_{ij}d\varepsilon_{ij}$, the equation 47 becomes

$$\begin{aligned} dF &= 2\mu\sum_{ijl}\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}\right)d\varepsilon_{ij} + K\sum_{l}\varepsilon_{ll}\,\delta_{ij}d\varepsilon_{ij} \\ \frac{dF}{d\varepsilon_{ij}} &= 2\mu\sum_{l}(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}) + \sum_{l}K\varepsilon_{ll}\,\delta_{ij} \\ \sigma_{ij} &= 2\mu\sum_{l}(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}) + \sum_{l}K\varepsilon_{ll}\,\delta_{ij} \end{aligned} \tag{48}$$

The expression 48 determines the stress tensor in terms of the strain tensor for an isotropic body. For hydrostatic compression, $i = j$, then the equation 48 becomes

$$\begin{aligned}
\sum_{ij}\sigma_{ij} &= 2\mu\left(\sum_{ij}\varepsilon_{ij}-\frac{1}{3}\sum_{ijl}\delta_{ij}\varepsilon_{ll}\right)+\sum_{ijl}K\varepsilon_{ll}\,\delta_{ij}\\
\sum_{i}\sigma_{ii} &= 2\mu\left(\sum_{i}\varepsilon_{ii}-\sum_{l}\varepsilon_{ll}\right)+\sum_{l}3K\varepsilon_{ll}\\
\sum_{l}\sigma_{ll} &= \sum_{l}3K\varepsilon_{ll}\\
\varepsilon_{ll} &= \frac{\sigma_{ll}}{3K} \qquad (49)
\end{aligned}$$

On substituting equation 49 in 48,

$$\begin{aligned}
\sigma_{ij} &= 2\mu\varepsilon_{ij}-\frac{2\mu}{9K}\sum_{l}\sigma_{ll}\delta_{ij}+\sum_{l}\frac{\sigma_{ll}}{3}\,\delta_{ij}\\
2\mu\varepsilon_{ij} &= \sigma_{ij}+\frac{2\mu}{9K}\sum_{l}\sigma_{ll}\delta_{ij}-\sum_{l}\frac{\sigma_{ll}}{3}\,\delta_{ij}\\
\varepsilon_{ij} &= \frac{1}{9K}\sum_{l}\sigma_{ll}\delta_{ij}+\frac{1}{2\mu}\left(\sigma_{ij}-\frac{1}{3}\sum_{l}\sigma_{ll}\delta_{ij}\right) \qquad (50)
\end{aligned}$$

Thus, the strain tensor ε_{ij} is a linear function of the stress tensor σ_{ij}. That is, under small deformation the strain is proportional to the applied forces, is called *Hooke's law.*

On multiplying ε_{ij} to equation 48,

$$\varepsilon_{ij}\sigma_{ij} = 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\sum_l \varepsilon_{ll}\right)\varepsilon_{ij} + K\sum_l \varepsilon_{ll}\,\delta_{ij}\varepsilon_{ij} \tag{51}$$

By taking $i = j$ in hydrostatic compression terms (second term), the equation 51 becomes

$$\varepsilon_{ij}\sigma_{ij} = 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\sum_l \varepsilon_{ll}\right)\varepsilon_{ij} + K\sum_l \varepsilon_{ll}^2 \tag{52}$$

On substituting for ε_{ij} using equation 44,

$$\varepsilon_{ij}\sigma_{ij} = 2\mu\left[\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\sum_l \varepsilon_{ll}\right)^2 + \left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\sum_l \varepsilon_{ll}\right)\frac{1}{3}\delta_{ij}\sum_l \varepsilon_{ll}\right] + K\sum_l \varepsilon_{ll}^2$$

$$\varepsilon_{ij}\sigma_{ij} = 2\mu\left[\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\sum_l \varepsilon_{ll}\right)^2 + \frac{1}{3}\varepsilon_{ij}\delta_{ij}\sum_l \varepsilon_{ll} - \frac{1}{9}\delta_{ij}^2\sum_l \varepsilon_{ll}^2\right] + K\sum_l \varepsilon_{ll}^2 \tag{53}$$

By taking $i = j$, $\sum_{ijl} \frac{1}{3}\varepsilon_{ij}\delta_{ij}\varepsilon_{ll} - \frac{1}{9}\sum_{ijl}\delta_{ij}^2\varepsilon_{ll}^2 = 0$, the equation 53 becomes

$$\varepsilon_{ij}\sigma_{ij} = 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\sum_{l}\varepsilon_{ll}\right)^2 + K\sum_{l}\varepsilon_{ll}^2 \qquad (54)$$

On comparing equation 46 and 53, we get

$$\sum_{ij}\varepsilon_{ij}\sigma_{ij} = 2F$$

$$F = \frac{1}{2}\sum_{ij}\varepsilon_{ij}\sigma_{ij} \qquad (55)$$

The expression 55 gives the free energy interms of stress and strain.

3.2. Homogeneous deformations

Homogeneous deformation is the one in which the strain tensor is constant throughout the volume of the body. Let us consider a simple extension (or compression) of a rod. Let the rod be along the z-axis, and let forces be applied to its ends which stretch it. There is no external force on the sides of the rod, so that all the components σ_{ik} except σ_{zz} are zero. If P be the force per unit area, then $\sigma_{zz} = P$. The component of the strain tensor in terms of the stress tensor is given by

$$\varepsilon_{ij} = \frac{1}{9K}\sum_l \sigma_{ll}\delta_{ij} + \frac{1}{2\mu}\left(\sigma_{ij} - \frac{1}{3}\sum_l \sigma_{ll}\delta_{ij}\right) \tag{56}$$

$$\varepsilon_{xx} = \frac{1}{9K}\sum_l \sigma_{ll} + \frac{1}{2\mu}\left(\sigma_{xx} - \frac{1}{3}\sum_l \sigma_{ll}\right)$$

$$\varepsilon_{xx} = \frac{1}{9K}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}) + \frac{1}{2\mu}\left[\sigma_{xx} - \frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\right]$$

Since $\sigma_{xx} = 0, \sigma_{yy} = 0$ and $\sigma_{zz} = P$,

$$\varepsilon_{xx} = \frac{1}{9K}P - \frac{1}{6\mu}P$$

$$\varepsilon_{xx} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right)P \tag{57}$$

Similarly,

$$\varepsilon_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3K}\right)P \tag{58}$$

From equation 56,

$$\begin{aligned}\varepsilon_{zz} &= \frac{1}{9K}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) + \frac{1}{2\mu}\left[\sigma_{zz} - \frac{1}{3}\left(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\right)\right] \\ &= \frac{1}{9K}P + \frac{1}{3\mu}P \\ \varepsilon_{zz} &= \frac{1}{3}\left(\frac{1}{\mu}+\frac{1}{3K}\right)P \qquad (59)\end{aligned}$$

The component ε_{zz} gives the relative lengthening of the rod. The coefficient of P is called the *coefficient of extension*, and its reciprocal is the *modulus of extension* or *Young's modulus*, E and is given by

$$\varepsilon_{zz} = \frac{P}{E} \qquad (60)$$

On comparing equations 60 and 59,

$$E = \frac{9\mu K}{3K+\mu} \qquad (61)$$

The components ε_{xx} and ε_{yy} give the relative compression of the rod in the transverse direction. The ratio of the transverse compression to the longitudinal extension is called *Poisson's ratio*, σ and is given by

$$\begin{aligned}
\varepsilon_{xx} = \varepsilon_{yy} &= -\sigma\varepsilon_{zz} \\
\sigma &= -\frac{\varepsilon_{xx}}{\varepsilon_{zz}} \\
\sigma &= \frac{1}{2}\frac{3K - 2\mu}{3K + \mu} \qquad (62)
\end{aligned}$$

When $K = 0$, $\sigma = -1$ and when $\mu = 0, \sigma = \frac{1}{2}$, Thus

$$-1 \leq \sigma \leq \frac{1}{2} \qquad (63)$$

The relative increase in volume of the rod is

$$\sum_i \varepsilon_{ii} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

By using equation 57, 58 and 59, we get

$$\sum_i \varepsilon_{ii} = \frac{P}{3K} \qquad (64)$$

Free energy of the body is given by

$$F = \frac{1}{2}\varepsilon_{zz}\sigma_{zz} = \frac{P^2}{E} \qquad \text{According to equation 60} \qquad (65)$$

From equation 61, we can get

$$\mu = \frac{3KE}{9K - E} \tag{66}$$

On substituting 66 to equation 62,

$$K = \frac{E}{3(1-2\sigma)} \tag{67}$$

On substituting 67 to equation 66,

$$\mu = \frac{E}{2(1+\sigma)} \tag{68}$$

Since,

$$\begin{aligned} K &= \lambda + \frac{2}{3}\mu \\ \lambda &= \frac{E}{3(1-2\sigma)} - \frac{E}{3(1+\sigma)} \\ \lambda &= \frac{E\sigma}{(1-2\sigma)(1+\sigma)} \end{aligned} \tag{69}$$

On substituting for μ and K by using equations 68 and 67 to the equation 56, the strain tensor can be written as

$$\begin{aligned}\varepsilon_{ij} &= \frac{(1-2\sigma)}{3E}\sum_l \sigma_{ll}\delta_{ij} + \frac{(1+\sigma)}{E}\left(\sigma_{ij} - \frac{1}{3}\sum_l \sigma_{ll}\delta_{ij}\right)\\ \varepsilon_{ij} &= \frac{1}{E}\left[(1+\sigma)\sigma_{ij} - \sum_l \sigma_{ll}\delta_{ij}\right] \qquad (70)\end{aligned}$$

The stress tensor is given by

$$\sigma_{ij} = 2\mu\sum_l(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}) + \sum_l K\varepsilon_{ll}\,\delta_{ij}$$

On substituting for μ and K by using equations 68 and 67,

$$\begin{aligned}\sigma_{ij} &= \frac{E}{(1+\sigma)}\sum_l(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}) + \sum_l \frac{E}{3(1-2\sigma)}\varepsilon_{ll}\,\delta_{ij}\\ \sigma_{ij} &= \frac{E}{(1+\sigma)}\left(\varepsilon_{ij} + \frac{\sigma}{(1-2\sigma)}\sum_l \varepsilon_{ll}\,\delta_{ij}\right) \qquad (71)\end{aligned}$$

Free energy is given by

$$\begin{aligned} F &= F_o + \frac{1}{2}\sum_{ij} C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \\ \frac{\delta F}{\delta \varepsilon_{ij}} &= C_{ijkl}\varepsilon_{kl} \\ \sigma_{ij} &= C_{ijkl}\varepsilon_{kl} \\ C_{ijkl} &= \frac{\sigma_{ij}}{\varepsilon_{kl}} \end{aligned} \tag{72}$$

Equation 71 in 72 gives

$$C_{ijkl} = \frac{E}{(1+\sigma)}\left(\frac{\varepsilon_{ij}}{\varepsilon_{kl}} + \frac{\sigma}{(1-2\sigma)}\sum_{l}\frac{\varepsilon_{ll}}{\varepsilon_{kl}}\delta_{ij}\right) \tag{73}$$

Thus the elastic modulus tensor C_{ijkl} can be expressed in terms of strain tensor. The tensor C_{ijkl} is a tensor of rank 4 and has 81 elements. It is a doubly symmetric tensor; $C_{ijkl} = C_{jikl}$, $C_{ijkl} = C_{ijlk}$ and $C_{ijkl} = C_{jilk}$.

For an isotropic material, the elements of elastic modulus tensor can be calculated by using equation 73.

$$C_{1111} = C_{2222} = C_{3333} = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)}$$

$$C_{1122} = C_{1133} = C_{2233} = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$$

$$C_{1212} = C_{1313} = C_{2323} = \frac{E}{1+\sigma}$$

4. The equations of equilibrium for isotropic bodies

In equilibrium the internal stresses in every volume element must balance, i.e. we must have $\sum_i F_i = 0$. Thus the equations of equilibrium for a deformed body are

$$\sum_i F_i = \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} = 0 \tag{74}$$

If the body is in a gravitational field, then

$$\sum_i F_i = \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} + \rho \sum_i g_i = 0 \tag{75}$$

where ρ is the density and g is the gravitational acceleration vector, directed vertically downwards.

On substituting for σ_{ij} by using equation 71,

$$\frac{E}{(1+\sigma)}\left(\sum_{ij}\frac{\delta\varepsilon_{ij}}{\delta x_j}+\frac{\sigma}{(1-2\sigma)}\sum_{ijl}\frac{\delta\varepsilon_{ll}}{\delta x_j}\delta_{ij}\right)+\rho\sum_i g_i \;=\; 0$$

$$\frac{E}{(1+\sigma)}\left(\sum_{ij}\frac{\delta\varepsilon_{ij}}{\delta x_j}+\frac{\sigma}{(1-2\sigma)}\sum_{il}\frac{\delta\varepsilon_{ll}}{\delta x_i}\right)+\rho\sum_i g_i \;=\; 0 \qquad (76)$$

Since,

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\delta u_i}{\delta x_j}+\frac{\delta u_j}{\delta x_i}\right) \qquad \text{(refer equation 7)} \qquad (77)$$

Substituting equation 77 in equation 76,

$$\frac{E}{2(1+\sigma)}\left[\sum_{ij}\left(\frac{\delta^2 u_i}{\delta x_j^2}+\frac{\delta^2 u_j}{\delta x_j\delta x_i}\right)+\frac{\sigma}{(1-2\sigma)}\sum_{il}\left(\frac{\delta^2 u_l}{\delta x_i\delta x_l}+\frac{\delta^2 u_l}{\delta x_i\delta x_l}\right)\right]+\rho\sum_i g_i \;=\; 0$$

Since,

$$\frac{\delta^2 u_j}{\delta x_j\delta x_i}=\frac{\delta^2 u_l}{\delta x_i\delta x_l}$$

$$
\begin{aligned}
\frac{E}{2(1+\sigma)}\left[\sum_{ij}\frac{\delta^2 u_i}{\delta x_j^2}+\left(1+\frac{2\sigma}{1-2\sigma}\right)\sum_{il}\frac{\delta^2 u_l}{\delta x_i \delta x_l}\right]+\rho\sum_i g_i &= 0 \\
\frac{E}{2(1+\sigma)}\sum_{ij}\frac{\delta^2 u_i}{\delta x_j^2}+\frac{E}{2(1+\sigma)(1-2\sigma)}\sum_{il}\frac{\delta^2 u_l}{\delta x_i \delta x_l}+\rho\sum_i g_i &= 0 \qquad (78)
\end{aligned}
$$

Since

$$
\frac{E}{2(1+\sigma)} = \mu, \qquad \frac{E\sigma}{(1+\sigma)(1-2\sigma)} = \lambda
$$

$$\mu \sum_{ij} \frac{\delta^2 u_i}{\delta x_j^2} + \frac{\lambda}{2\sigma} \sum_{il} \frac{\delta}{\delta x_i} \frac{\delta u_l}{\delta x_l} + \rho \sum_i g_i = 0$$

$$(\bigtriangledown^2 u_i) + \frac{\lambda}{2\sigma\mu} \sum_{il} \frac{\delta}{\delta x_i} (\bigtriangledown.\mathbf{u}) + \frac{\rho}{\mu} \sum_i g_i = 0$$

$$\begin{aligned} \bigtriangledown^2 (iu_x + ju_y + ku_z)(i+j+k) + \\ \frac{\lambda}{2\sigma\mu} \left(i\frac{\delta}{\delta x} + j\frac{\delta}{\delta y} + k\frac{\delta}{\delta z} \right) (i+j+k)(\bigtriangledown.\mathbf{u}) + \\ \frac{\rho}{\mu} (ig_x + jg_y + kg_z)(i+j+k) &= 0 \end{aligned} \tag{79}$$

$$\bigtriangledown^2 \mathbf{u} + \frac{\lambda}{2\sigma\mu} \bigtriangledown (\bigtriangledown.\mathbf{u}) + \frac{\rho}{\mu}\mathbf{g} = 0$$

$$\operatorname{div} \mathbf{grad}\, \mathbf{u} + \frac{\lambda}{2\sigma\mu} \mathbf{grad} \operatorname{div} \mathbf{u} + \frac{\rho}{\mu}\mathbf{g} = 0$$

$$\triangle\, \mathbf{u} + \frac{\lambda}{2\sigma\mu} \mathbf{grad} \operatorname{div} \mathbf{u} + \frac{\rho}{\mu}\mathbf{g} = 0 \tag{80}$$

where $\triangle \equiv \operatorname{div} \mathbf{grad}\, \mathbf{u}$. If the deformation of the body is caused, not by body forces, but by forces applied to its surface only, $\mathbf{g} = 0$, then equations of equilibrium can be written as

$$\triangle\ \mathbf{u} + \frac{\lambda}{2\sigma\mu}\,\mathbf{grad}\,\mathrm{div}\,\mathbf{u} \;=\; 0 \tag{81}$$

4.1. Navier's equations

Flow induced displacement of solid bodies are generally sufficiently small to be described by the linearized equations of elasticity and are called Navier's equations. The stress tensor σ_{ij} in terms of strain tensor ε_{ij} is given by

$$\begin{aligned}
\sigma_{ij} &= 2\mu \sum_l (\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{ll}) + \sum_l K\varepsilon_{ll}\,\delta_{ij} \\
&= 2\mu\varepsilon_{ij} - \frac{2\mu}{3}\sum_l \delta_{ij}\varepsilon_{ll} + \sum_l K\varepsilon_{ll}\,\delta_{ij} \\
&= 2\mu\varepsilon_{ij} - \frac{2\mu}{3}\sum_l \varepsilon_{ll}\delta_{ij} + \sum_l K\varepsilon_{ll}\,\delta_{ij} \qquad (82)
\end{aligned}$$

Since,

$$\mu = \frac{E}{2(1+\sigma)}, \qquad K = \frac{E}{3(1-2\sigma)} \qquad \text{and} \qquad \lambda = \frac{\mathrm{E}\sigma}{(1-2\sigma)(1+\sigma)} \tag{83}$$

Substituting for μ and K from equation 83 to equation 82,

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda \sum_l \varepsilon_{ll}\, \delta_{ij}$$

$$\frac{\delta\sigma_{ij}}{\delta x_j} = 2\mu\frac{\delta\varepsilon_{ij}}{\delta x_j} + \lambda \sum_l \frac{\delta\varepsilon_{ll}}{\delta x_j}\, \delta_{ij} \tag{84}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\delta u_i}{\delta x_j} + \frac{\delta u_j}{\delta x_i}\right), \qquad \varepsilon_{ll} = \frac{\delta u_l}{\delta x_l}$$

$$\frac{\delta\varepsilon_{ij}}{\delta x_j} = \frac{1}{2}\left(\frac{\delta^2 u_i}{\delta x_j^2} + \frac{\delta^2 u_j}{\delta x_j \delta x_i}\right), \qquad \frac{\delta\varepsilon_{ll}}{\delta x_j} = \frac{\delta^2 u_l}{\delta x_j \delta x_l} \tag{85}$$

Equation 85 in 84 gives

$$\frac{\delta\sigma_{ij}}{\delta x_j} = \mu\left(\frac{\delta^2 u_i}{\delta x_j^2} + \frac{\delta^2 u_j}{\delta x_j \delta x_i}\right) + \lambda \sum_l \frac{\delta^2 u_l}{\delta x_j \delta x_l} \tag{86}$$

Since,

$$\frac{\delta^2 u_j}{\delta x_j \delta x_i} = \frac{\delta^2 u_l}{\delta x_j \delta x_l}$$

$$\frac{\delta\sigma_{ij}}{\delta x_j} = \mu\frac{\delta^2 u_i}{\delta x_j^2} + (\mu + \lambda)\frac{\delta^2 u_j}{\delta x_j \delta x_i} \tag{87}$$

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The equations of equilibrium for a deformed body is given by

$$\sum_i F_i = \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} + \rho\, g$$

$$\rho \sum_i \frac{\delta^2 u_i}{\delta t^2} = \sum_{ij} \frac{\delta \sigma_{ij}}{\delta x_j} + \rho\, g \tag{88}$$

Equation 87 in equation 88 gives

$$\rho \sum_i \frac{\delta^2 u_i}{\delta t^2} = \mu \sum_{ij} \frac{\delta^2 u_i}{\delta x_j^2} + (\mu + \lambda) \frac{\delta^2 u_j}{\delta x_j \delta x_i} + \rho\, g \tag{89}$$

The equation 89 in cartesian components can be written as

$$\frac{\delta^2 u_x}{\delta t^2} = \mu \left(\frac{\delta^2 u_x}{\delta x^2} + \frac{\delta^2 u_x}{\delta y^2} + \frac{\delta^2 u_x}{\delta z^2} \right) + (\mu + \lambda) \left(\frac{\delta^2 u_x}{\delta x^2} + \frac{\delta^2 u_y}{\delta y \delta x} + \frac{\delta^2 u_z}{\delta z \delta x} \right) + \rho g \tag{90}$$

$$\frac{\delta^2 u_y}{\delta t^2} = \mu \left(\frac{\delta^2 u_y}{\delta x^2} + \frac{\delta^2 u_y}{\delta y^2} + \frac{\delta^2 u_y}{\delta z^2} \right) + (\mu + \lambda) \left(\frac{\delta^2 u_x}{\delta x \delta y} + \frac{\delta^2 u_y}{\delta y^2} + \frac{\delta^2 u_z}{\delta z \delta y} \right) + \rho g \tag{91}$$

$$\frac{\delta^2 u_z}{\delta t^2} = \mu \left(\frac{\delta^2 u_z}{\delta x^2} + \frac{\delta^2 u_z}{\delta y^2} + \frac{\delta^2 u_z}{\delta z^2} \right) + (\mu + \lambda) \left(\frac{\delta^2 u_x}{\delta x \delta z} + \frac{\delta^2 u_y}{\delta y \delta z} + \frac{\delta^2 u_z}{\delta z^2} \right) + \rho g \tag{92}$$

Equation 90, 91 and 92 are the Navier equations of motion in Cartesian coordinates.

Again consider equation 89

$$\rho \sum_i \frac{\delta^2 u_i}{\delta t^2} = \mu \sum_{ij} \frac{\delta^2 u_i}{\delta x_j^2} + (\mu + \lambda) \frac{\delta^2 u_j}{\delta x_j \delta x_i} + \rho\, g$$

$$= \mu \left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2} \right) \sum_i u_i + (\mu + \lambda) \sum_{ij} \frac{\delta}{\delta x_j} \frac{\delta u_i}{\delta x_i} + \rho\, g$$

$$\rho \frac{\delta^2 \mathbf{u}}{\delta t^2} = \mu \bigtriangledown^2 \mathbf{u} + (\mu + \lambda) \bigtriangledown (\bigtriangledown . \mathbf{u}) + \rho\, g \qquad (93)$$

where $\mathbf{u} = iu_x + ju_y + ku_z$. The expression 93 gives the Navier equations in vector form.

5. Fluid Mechanics

A fluid is a substance in which the constituent molecules are free to move relative to each other.

Fluid mechanics is the study of how fluids move and the forces on them. (Fluids include liquids and gases.) Fluid mechanics can be divided into fluid statics, the study of fluids at rest, and fluid dynamics, the study of fluids in motion.

Consider a region of space with volume V bounded by the surface S, be fixed with respect to time. Let $\rho = \rho(x, y, z, t)$ be the fluid density at any point (x, y, z) of the fluid in volume V at any time t. A fluid continuum, like a solid continuum, is characterized by conservation laws describing:

1. Conservation of linear momentum

$$\rho \dot{v}_i = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g \tag{94}$$

2. Conservation of angular momentum

$$\sigma_{ij} = \sigma_{ji} \tag{95}$$

where σ_{ij} is the stress tensor, ρ is density of the fluid and g is the gravitational acceleration vector, directed vertically downwards.

3. Conservation of mass (continuity equation)

$$\text{Rate of change} \;+\; \text{advection} \;+\; \text{diffusion} \;=\; \text{source}$$
$$\frac{\partial}{\partial t}\int_V \rho\, dV \;+\; \oint_S \rho\,(\vec{v}.\hat{n})dS + \frac{1}{\vec{v}}\oint_S (\sigma_{ij}\,.\hat{n})dS = \frac{1}{\vec{v}}\oint_S P.\hat{n}dS \quad (96)$$

where

$\frac{\partial}{\partial t}\int_V \rho\, dV$ is the total rate of mass increase within V,

$\oint_S \rho\,(\vec{v}.\hat{n})\, dS$ is the rate of mass flow in to V and $\vec{v}$ is the velocity of the fluid,

$\frac{1}{\vec{v}}\oint_S (\sigma_{ij}\,.\hat{n})dS$ is the rate of mass increase due to stress tensor σ_{ij},

and $\frac{1}{\vec{v}}\oint_S P.\hat{n}dS$ is the rate of mass of the fluid flow in to volume V.

In the absence of source and if there is no internal stresses, the equation 96, becomes

$$\int_V \frac{\partial \rho}{\partial t}\, dV \;+\; \oint_S \hat{n}\,.\,(\rho\vec{v})dS = 0$$

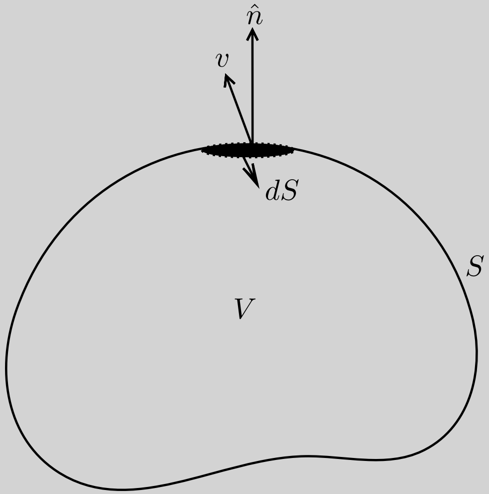

Figure 3:

According to the Gauss divergence theorem,

$$\oint_S \hat{n}\,.\,(\rho\vec{v})dS = \int_V \nabla\,.\,(\rho\vec{v})\,dV$$

$$\int_V \frac{\partial\rho}{\partial t}\,dV + \int_V \nabla\,.\,(\rho\vec{v})\,dV = 0$$

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\,.\,(\rho\vec{v})\right] dV = 0 \tag{97}$$

Since equation 97 must hold for each volume element dV, the integrand must vanish each point inside V. That is,

$$\frac{\partial \rho}{\partial t} + \nabla \, . \, (\rho \vec{v}) \;=\; 0 \tag{98}$$

Equation 98 is called equation of continuity which must hold at any point of fluid free from sources and sinks.

If motion is steady, then $\frac{\partial \rho}{\partial t} = 0$, then $\nabla \, . \, (\rho \vec{v}) \;= 0$. That is, if the fluid is incompressible, ρ is constant throughout the fluid.

On expanding equation 98,

$$\frac{\partial \rho}{\partial t} + (\nabla \rho) . \vec{v} + \rho \, (\nabla . \vec{v}) \;=\; 0 \tag{99}$$

$$\frac{D\rho}{Dt} + \rho \, (\nabla . \vec{v}) \;=\; 0 \tag{100}$$

where

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\nabla \rho) . \vec{v}$$

$\frac{D}{Dt}$ is called Lagrangian time derivative.

The strain tensor is given by

$$
\begin{aligned}
\varepsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \\
\frac{d\varepsilon_{ij}}{dt} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\
\varepsilon_{ij} &= \frac{1}{2}\int\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dt \\
\varepsilon_{ij} &= \int D_{ij}\, dt \qquad (101)
\end{aligned}
$$

where

$$
D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (102)
$$

is called the *rate of deformation tensor*, *velocity strain tensor*, or *rate of strain tensor*.

6. Navier-Stokes equations

The Navier-Stokes equations are the set of equations that describe the motion of fluid substances such as liquids and gases. These equations state that changes in momentum (force) of fluid particles depend only on the external pressure and internal viscous forces (similar to friction) acting on the fluid. Thus, the Navier-Stokes equations describe the balance of forces acting at any given region of the fluid.

According to conservation of linear momentum, the general equation of motion for any fluid is given by

$$\begin{aligned} \rho \sum_i \dot{v}_i &= \sum_{ij} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g \\ \rho \sum_i \left(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j} \frac{\partial x_j}{\partial t} \right) &= \sum_{ij} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g \\ \rho \sum_i \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) &= \sum_{ij} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g \qquad (103) \end{aligned}$$

where σ_{ij} is the stress tensor and is given by

$$\sigma_{ij} = -P\delta_{ij} \quad \text{for fluid is at rest} \tag{104}$$

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij} \quad \text{for fluid is in motion} \tag{105}$$

where P is the force per unit area and τ_{ij} is called the viscous stress tensor. τ_{ij} in terms of rate of strain tensor D_{ij} is given by

$$\tau_{ij} = 2\mu \sum_l (D_{ij} - \frac{1}{3}\delta_{ij} D_{ll}) + \sum_l \xi D_{ll}\, \delta_{ij} \quad \text{refer equation 48} \tag{106}$$

where $\xi = (\lambda + \frac{2}{3}\mu)$ is called *coefficient of bulk viscosity*. λ and μ are the Lamé coefficients. $\eta = \mu$ is called *coefficient of shearing viscosity*. Then,

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

$$\sigma_{ij} = -P\delta_{ij} + 2\eta \sum_l (D_{ij} - \frac{1}{3}\delta_{ij} D_{ll}) + \sum_l \xi D_{ll}\, \partial_{ij}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial P}{\partial x_j}\delta_{ij} + 2\eta \sum_l \left(\frac{\partial D_{ij}}{\partial x_j} - \frac{1}{3}\delta_{ij}\frac{\partial D_{ll}}{\partial x_j} \right) + \sum_l \xi \frac{\partial D_{ll}}{\partial x_j}\, \delta_{ij}$$

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial P}{\partial x_j} + 2\eta \sum_l \left(\frac{\partial D_{ij}}{\partial x_j} - \frac{1}{3}\frac{\partial D_{ll}}{\partial x_j} \right) + \sum_l \xi \frac{\partial D_{ll}}{\partial x_j} \tag{107}$$

Substituting for D_{ij} by using equation 102,

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial P}{\partial x_j} + \eta \frac{\partial^2 v_i}{\partial x_j^2} + \eta \frac{\partial^2 v_j}{\partial x_i \partial x_j} - \frac{2\eta}{3} \sum_l \frac{\partial^2 v_l}{\partial x_j \partial x_l} + \xi \sum_l \frac{\partial^2 v_l}{\partial x_j \partial x_l} \tag{108}$$

Since,

$$\frac{\partial^2 v_j}{\partial x_i \partial x_j} = \frac{\partial^2 v_l}{\partial x_j \partial x_l}$$

equation 108 becomes

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial P}{\partial x_j} + \eta \frac{\partial^2 v_i}{\partial x_j^2} + \left(\xi + \frac{\eta}{3}\right) \frac{\partial^2 v_j}{\partial x_i \partial x_j} \tag{109}$$

On substituting equation 109 in equation 103,

$$\rho \sum_i \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = -\sum_j \frac{\partial P}{\partial x_j} + \eta \sum_{ij} \frac{\partial^2 v_i}{\partial x_j^2} + \left(\xi + \frac{\eta}{3}\right) \sum_{ij} \frac{\partial^2 v_j}{\partial x_i \partial x_j} + \rho g$$

$$\rho \sum_i \left(\frac{\partial v_i}{\partial t} + v_j . \frac{\partial}{\partial x_j} v_j\right) = -\sum_j \frac{\partial P}{\partial x_j} + \eta \sum_i \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v_i$$

$$+ \left(\xi + \frac{\eta}{3}\right) \sum_{ij} \frac{\partial}{\partial x_i} \frac{\partial v_j}{\partial x_j} - \rho \sum_i \frac{\partial \Omega}{\partial x_i}$$

where Ω is the conservative body force potential given by $g = -\nabla\Omega$.

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v}\right] = -\sum_j \frac{\partial P}{\partial x_j} + \eta\sum_i \nabla^2 v_i + \left(\xi + \frac{\eta}{3}\right)\sum_i \frac{\partial}{\partial x_i}(\nabla.\vec{v}) - \rho\sum_i \frac{\partial\Omega}{\partial x_i}$$

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\nabla)\vec{v} = -\frac{1}{\rho}\nabla P + \frac{\eta}{\rho}\nabla^2\vec{v} + \frac{1}{\rho}\left(\xi + \frac{\eta}{3}\right)\nabla(\nabla.\vec{v}) - \nabla\Omega$$

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}.\mathbf{grad})\vec{v} = -\frac{1}{\rho}\mathbf{grad}P + \frac{\eta}{\rho}\triangle\vec{v} + \frac{1}{\rho}\left(\xi + \frac{\eta}{3}\right)\mathbf{grad}(\nabla.\vec{v}) - \nabla\Omega \qquad (110)$$

The equation 110 is called Navier-Stokes' equation. For incompressible fluid

$$\varepsilon_{ii} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = 0 \qquad (111)$$

$$\int D_{ii} = \int \frac{\partial v_i}{\partial x_i} = \int \nabla.\vec{v} = 0 \qquad (112)$$

That is, if fluid is incompressible, $\nabla.\vec{v} = 0$, then the equation 110 becomes

$$\frac{\partial\vec{v}}{\partial t} + (\nabla.\vec{v})\vec{v} = -\frac{1}{\rho}\nabla P + \frac{\eta}{\rho}\nabla^2\vec{v} - \nabla\Omega \qquad (113)$$

The equation 113 gives Navier-Stokes' equation for incompressible fluids.

7. Flow through a cylindrical pipe: The Poiseuille formula

The rate at which an incompressible viscous fluid flows through a cylindrical pipe can be calculated from the Navier-Stokes equation. The result is called Poiseuille's Law.

Let us consider the velocity field to be zero at the boundaries, where it touches the walls of the pipe. Just inside the boundary, the steady state velocity to be same, all the way around the circumference. That is the velocity field to have circular symmetry, with surfaces of equal velocity being cylinders parallel to the axis of the pipe.

The Navier-stokes equation for flow of incompressible fluid is given by

$$\frac{\partial \vec{v}}{\partial t} + (\nabla.\vec{v})\vec{v} \; = \; -\frac{1}{\rho}\nabla P + \frac{\eta}{\rho}\nabla^2\vec{v} - g \tag{114}$$

From the figure it is clear that $v = (v_x, 0, 0)$. For a steady state solution,

$$\frac{\partial \vec{v}}{\partial t} = 0$$

Since the streamlines have constant velocity, there is no acceleration associated with change of position of volume element. Therefore $(\nabla.\vec{v})\vec{v} = 0$. It is assumed that the gravitational effects are small, and set $g = 0$. Then the equation 114 becomes

$$-\frac{1}{\rho}\nabla P + \frac{\eta}{\rho}\nabla^2 v_x \;=\; 0 \qquad (115)$$

In cylindrical coordinates, the equation 115 can be written as

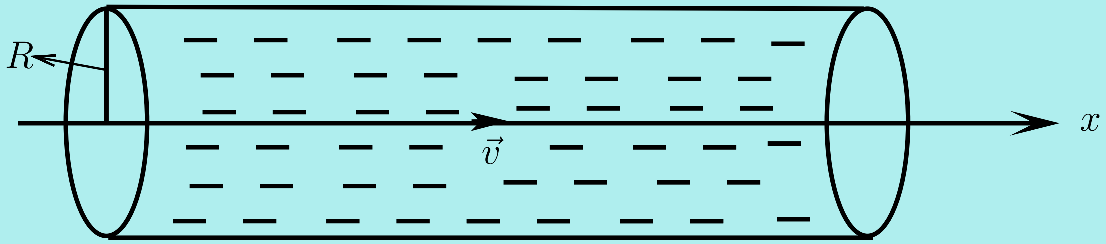

Figure 4: Flow of a viscous fluid in a cylindrical tube.

$$-\left(\frac{\partial P}{\partial r} + \frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{\partial P}{\partial x}\right) + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_x}{\partial \theta^2} + \frac{\partial^2 v_x}{\partial x^2}\right] \;=\; 0 \qquad (116)$$

The assumption of cylindrical symmetry, plus the assumption that the (symmetry-breaking) gravitational field is zero, means that our fields must be independent of the angle θ, so

$$\frac{\partial P}{\partial \theta} = 0 \qquad \text{and} \qquad \frac{\partial v_x}{\partial \theta} = 0$$

Since, $v = (v_x, 0, 0)$ is constant along x direction,

$$\frac{\partial v_x}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial P}{\partial r} = 0$$

The equation 116, then reduces to

$$\frac{\partial P}{\partial x} = \frac{\eta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) \tag{117}$$

For steady flow $\frac{\partial P}{\partial x} = constant = -\alpha\ (say)$, then the equation 117 can be written as

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) = -\frac{\alpha r}{\eta}$$

$$r\frac{\partial v_x}{\partial r} = -\int \frac{\alpha r}{\eta} dr$$

$$r\frac{\partial v_x}{\partial r} = -\frac{\alpha r^2}{2\eta} + a \tag{118}$$

$$v_x = -\int \frac{\alpha r}{2\eta} dr + \int \frac{a}{r}$$

$$v_x = -\frac{\alpha r^2}{4\eta} + a \log r + b \tag{119}$$

where a and b are constants, can be evaluated using boundary conditions. At center of the tube $r = 0$ and v_x is maximum. The equation 118 yields $a = 0$. At edge of the tube $r = R$ and $v_x = 0$. The equation 119 yields $b = \alpha R^2/4\eta$. Then the equation 119 becomes

$$v_x = \frac{\alpha}{4\eta}(R^2 - r^2)dr \tag{120}$$

If ρ is density of the tube, mass of fluid passes per unit time through an annular element $2\pi r\, dr$ is $\rho\, 2\pi r\, v_x\, dr$. Then the flow rate Q, the mass of fluid passes per unit time through any cross section of the tube (called the discharge) is

$$Q = 2\pi\rho \int_0^R r v_x\, dr \tag{121}$$

On substituting for v_x by using equation 120 to 121,

$$Q = \frac{\pi\rho\alpha}{2\eta}\int_0^R (R^2 r - r^3)dr$$

$$Q = \frac{\pi\rho\alpha R^4}{8\eta} \tag{122}$$

If ΔP is the pressure difference between any two transverse vertical sections separated by a distance l, then

$$\alpha = \frac{\partial P}{\partial x} = \frac{\Delta P}{l}$$

The equation 122 becomes

$$Q = \frac{\pi \rho R^4 \Delta P}{8 \eta l} \quad (123)$$

Equation 123 is called the Poiseuille formula.