Hamilton's equations

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1. The Hamilton equations of motion

Lagrange formulation is in terms of generalized coordinates q_i and generalized velocities $\dot{q}_i$ gives equations of motion, which are second order in time. Instead if we regard N generalized coordinates q_i and N generalized momenta p_i as independent variables, and again $q(t)$ and $p(t)$ at every instant of time t, we will get $2N$ first order equations. Hence the $2N$ equations of motion describe the behaviour of the system ina phase space whose coodinates are the $2N$ independent variables. These are called *canonical coordinates* and *canonical momenta*. This new formulation is by the Hamiltonian and is known as Hamiltonian formulation.

The Lagrange equations for a free particle can be written as

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{q}_i} - \frac{\delta L}{\delta q_i} = 0 \qquad (1)$$

where

$$L(q, \dot{q}, t) = T - V = \frac{1}{2}\sum_i m_i \dot{q}_i^{\,2} - V$$

$$\frac{\delta L}{\delta \dot{q}_i} = m_i \dot{q}_i = p_i \tag{2}$$

p_i are called generalized or conjugate momenta. Equation 2 in 1 gives,

$$\frac{dp_i}{dt} - \frac{\delta L}{\delta q_i} = 0$$

$$\dot{p}_i = \frac{\delta L}{\delta q_i} \tag{3}$$

The differential of the Lagrangian can be written as

$$dL = \sum_i \frac{\delta L}{\delta q_i} dq_i + \sum_i \frac{\delta L}{\delta \dot{q}_i} d\dot{q}_i + \frac{\delta L}{\delta t} dt \tag{4}$$

Equations 2 and 3 in 4,

$$dL = \sum_i \dot{p}_i \, dq_i + \sum_i p_i \, d\dot{q}_i + \frac{\delta L}{\delta t} \, dt \tag{5}$$

If we define the Hamiltonian $H(q,p,t)$ as a function of generalized coordinates q_i and generalized momenta p_i, the Legendre transformation generate the Hamiltonian

$$H(q,p,t) = \sum_i \dot{q}_i p_i - L(q,\dot{q},t) \tag{6}$$

The differential of the Hamiltonian is

$$\sum_i \frac{\delta H}{\delta q_i} dq_i + \sum_i \frac{\delta H}{\delta p_i} dp_i + \frac{\delta H}{\delta t} dt = \sum_i \dot{q}_i\, dp_i + \sum_i p_i\, d\dot{q}_i - dL \quad (7)$$

Equation 5 in 7,

$$\sum_i \frac{\delta H}{\delta q_i} dq_i + \sum_i \frac{\delta H}{\delta p_i} dp_i + \frac{\delta H}{\delta t} dt = \sum_i \dot{q}_i\, dp_i - \sum_i \dot{p}_i\, dq_i - \frac{\delta L}{\delta t} dt$$

$$\dot{q}_i = \frac{\delta H}{\delta p_i} \quad (8)$$

$$-\dot{p}_i = \frac{\delta H}{\delta q_i} \quad (9)$$

$$-\frac{\delta L}{\delta t} = \frac{\delta H}{\delta t} \quad (10)$$

Equations 8 and 9 are known as the canonical equations of Hamilton. They constitute the desired set of $2N$ first order equations of motion replacing the N second order Lagrange equations.

If (x, y, z) are the Cartesian coordinates at time t of a free material point of mass m moving in a potential field $V(x, y, z) = V(q_i)$, we may take $q_1 =$

$x, q_2 = y, q_3 = z$. The kinetic energy T is given by

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\sum_i \dot{q_i}^2$$

The Lagrangian for the particle is

$$\begin{aligned} T - V = L &= \frac{1}{2}m\sum_i \dot{q_i}^2 - V(q_i) \\ \frac{\delta L}{\delta \dot{q}_i} &= m\dot{q}_i \\ p_i &= m\dot{q}_i \end{aligned}$$

On substituting for L and p_i in the equation 6,

$$\begin{aligned} H &= \sum_i \dot{q}_i p_i - L = m\sum_i \dot{q_i}^2 - (T - V) \\ H &= T + V \end{aligned} \tag{11}$$

Thus the Hamiltonian becomes the total energy of the system.

1.1. Hamiltonian for a free particle in different coordinates

1. *Using Cartesian coordinates:* (x, y, z) are the Cartesian coordinates at time t of a free material point of mass m moving in a potential field $V(x, y, z)$. The kinetic energy T is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus the Hamiltonian for the particle is

$$\begin{aligned} T + V = H &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z) \\ H &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \end{aligned} \quad (12)$$

2. *Using cylindrical polar coordinates*: (r, θ, z) are the cylindrical coordinates at time t of a free material point of mass m in the potential field $V(r)$.

   The kinetic energy T is

$$\begin{aligned} T &= \frac{1}{2}m\left[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2\right] \\ &= \frac{1}{2m}\left[(m\dot{r})^2 + \frac{1}{r^2}(mr^2\dot{\theta})^2 + (m\dot{z})^2\right] \end{aligned}$$

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right]$$

Then

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right] + V(r)$$

3. *Using spherical polar coordinates*: (r, θ, ϕ) are the spherical polar coordinates at time t of a free material point of mass m in the potential field $V(r)$.

The kinetic energy T is

$$T = \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \left(r sin\theta\dot{\phi}\right)^2\right]$$

$$= \frac{1}{2m}\left[(m\dot{r})^2 + \frac{1}{r^2}\left(r^2\dot{\theta}\right)^2 + \frac{1}{r^2 sin^2\theta}\left(r^2 sin^2\theta\dot{\phi}\right)^2\right]$$

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right]$$

Then

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right] + V(r)$$

1.2. Hamiltonian for an electron in a Coulomb field

When an electron revolving about the charge e,

$$v(r) = -\frac{e^2}{r}$$

The kinetic energy T of electron in spherical coordinate is

$$T = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right]$$

Then

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 sin^2\theta}\right] - \frac{e^2}{r}$$

1.3. Hamiltonian for the simple harmonic oscillator

The Lagrangian for a simple harmonic oscillator can be written as

$$L = \frac{1}{2}m\sum_i \dot{q_i}^2 - \frac{1}{2}m\omega^2\sum_i {q_i}^2 = \sum_i \frac{p_i^2}{2m} - \frac{1}{2}m\omega^2\sum_i {q_i}^2$$

The cononical momentum is

$$p_i = \frac{\delta L}{\delta \dot{q}} = m\dot{q}_i$$
$$\dot{q}_i = \frac{p_i}{m}$$

Then

$$H = \sum_i p_i\dot{q}_i - L = \sum_i \frac{p_i^2}{m} - \sum_i \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 \sum_i {q_i}^2$$
$$H = \sum_i \frac{p_i^2}{2m} + \frac{1}{2}m\omega^2 \sum_i {q_i}^2$$

1.4. Hamiltonian for an electron in electromagnetic field

Consider a particle of mass m and charge e moving in an electromagnetic field. Lagrangian for the particle is

$$L = T - U = \frac{1}{2}m \sum_i {\dot{q}_i}^2 - e\left(\phi - \vec{A}.\sum_i \dot{q}_i\right)$$

where $e\left(\phi - \vec{A}.\dot{q}_i\right)$ is the velocity dependent potential.

$$\frac{\delta L}{\delta \dot{q}_i} = p_i = m\dot{q}_i + e\vec{A}$$
$$\dot{q}_i = \frac{1}{m}(p_i - e\vec{A})$$

The Hamiltonian H is

$$\begin{aligned}
H &= \sum_i p_i \dot{q}_i - L \\
&= m\sum_i \dot{q_i}^2 + e\vec{A}.\sum_i \dot{q}_i - \frac{1}{2}m\sum_i \dot{q_i}^2 + e\left(\phi - \vec{A}.\sum_i \dot{q}_i\right) \\
&= \frac{1}{2}m\sum_i \dot{q_i}^2 + e\phi \\
&= \frac{1}{2m}\left(\sum_i p_i - e\vec{A}\right)^2 + e\phi \\
H &= \frac{1}{2m}(\vec{p} - e\vec{A})^2 + e\phi
\end{aligned}$$

1.5. Cyclic coordinates

Consider a system of N degrees of freedom described by q_i generalized coordinates. The Lagrange's equations for the system are

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = 0$$

If Lagrangian of the system does not contain a given coordinate q_i even though it may contain corresponding velocity $\dot{q}_i$, then the coordinate q_i is said to be *cyclic or ignorable.* Then

$$\frac{\delta L}{\delta q_i} = 0$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) &= 0 \\ \frac{dp_i}{dt} &= 0 \\ p_i &= constant \end{aligned}$$

The generalized momentum conjugate to a cyclic coordinate is conserved.

Example: In a planetory motion, the angular momentum p_θ is constant

$p_\theta = mr^2\dot{\theta} = constant$. Here θ is cyclic.

1.6. Hamilton's Equations from a Variational Principle

The motion of a conservative system from its configuration at time t_1 to its configuration at time t_2 is such that the line integral between the time t_1 and t_2 of the Lagrangian of the system has a stationary value for the actual path of the motion.

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)dt = constant \tag{13}$$

$L = T - V$ is the Lagrangian. Since $\int L\,dt$ has the dimensions of $energy \times time$ called action, the principle is sometimes referred to as the principle of least action. The integral is called the action integral.

The variation of the action integral for fixed time t_1 and t_2 must be zero.

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\ \ dt = 0 \tag{14}$$

Byusing equation 6,

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\ \ dt \ = \ \delta \int_{t_1}^{t_2} \left(\sum_i \dot{q}_i p_i - H \right)\ dt = 0$$

$$= \int_{t_1}^{t_2} \left(\sum_i \dot{q}_i \delta p_i + \sum_i p_i \delta \dot{q}_i - \delta H \right) dt = 0 \qquad (15)$$

Since,

$$\delta H(q,p) = \sum_i \frac{\delta H}{\delta q_i}\delta q_i + \sum_i \frac{\delta H}{\delta p_i}\delta p_i \qquad (16)$$

Equation 16 in equation 15,

$$\sum_i \int_{t_1}^{t_2} \left(\dot{q}_i - \frac{\delta H}{\delta p_i} \right) \delta p_i \, dt + \int_{t_1}^{t_2} p_i \delta \dot{q}_i \, dt - \int_{t_1}^{t_2} \frac{\delta H}{\delta q_i}\delta q_i \, dt = 0$$

$$\sum_i \int_{t_1}^{t_2} \left(\dot{q}_i - \frac{\delta H}{\delta p_i} \right) \delta p_i \, dt + p_i \, \delta q_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \delta q_i \, \dot{p}_i \, dt - \int_{t_1}^{t_2} \frac{\delta H}{\delta q_i}\delta q_i \, dt = 0$$

Since the variation $\delta q_i = 0$ at the end point, the term $p_i \, dq_i \big|_{t_1}^{t_2} = 0$.

$$\sum_i \int_{t_1}^{t_2} \left(\dot{q}_i - \frac{\delta H}{\delta p_i} \right) \delta p_i \, dt - \int_{t_1}^{t_2} \left(\dot{p}_i + \frac{\delta H}{\delta q_i} \right) \delta q_i \, dt = 0$$

$$\sum_i \int_{t_1}^{t_2} \left[\left(\dot{q}_i - \frac{\delta H}{\delta p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\delta H}{\delta q_i} \right) \delta q_i \right] dt = 0$$

Since the system is holonomic and is described in the phase space, q_i's and p_i's are all independent, and δq_i's and δp_i's are orbitary at all points of the path.

The above integrals can vanish, only if

$$\dot{q}_i - \frac{\delta H}{\delta p_i} = 0 \quad \Longrightarrow \quad \dot{q}_i = \frac{\delta H}{\delta p_i}$$
$$\dot{p}_i + \frac{\delta H}{\delta q_i} = 0 \quad \Longrightarrow \quad -\dot{p}_i = \frac{\delta H}{\delta q_i}$$

which are Hamilton's eqiuations of motion.

2. Canonical Transformations

Canonical transformations are transformations of the coordinates and momenta (q, p) that preserve Hamilton's equations (though with a different Hamiltonian). The transformations from one set of coordinates q_i to a new set Q_i, by transformation equations of the form

$$Q_i = Q_i(q, t) \tag{17}$$

are called *point transformations*. It can be shown that under a point transformation, a system that obeys the Euler-Lagrange equations in the original coordinates continues to obey them in the new coordinates.

In the Hamiltonian formulation the momenta are also independent variables on the same level as the generalized coordinates. The concept of transformation of coordinates must therefore be widened to include the simultaneous transformation of the independent coordinates and momenta, (q_i, p_i), to a new set Q_i, P_i, with equations of transformation

$$Q_i = Q_i(q, p, t) \tag{18}$$

$$P_i = P_i(q, p, t) \tag{19}$$

These transformations are called *contact transformations*. An arbitrary contact transformation may not preserve Hamilton's equations. The transformations which preserve Hamilton's equations are known as *canonical transformations*.

Equations 17 define a *point transformation of configuration space* and, equations 18 and 19 define a *point transformation of phase space*.

$H(Q, P, t)$ is the Hamiltonian in the cananical coordinates and the equations of motion in the new cocodinates are in the Hamiltonian form

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} \qquad \dot{P}_i = -\frac{\delta H}{\delta Q_i} \tag{20}$$

The Hamilton principle in both old coordinates (q_i, p_i) and canonical coordinates (Q_i, P_i) are written as

$$\delta \int_{t_1}^{t_2} \left(\sum_i \dot{q}_i p_i - H(q, p, t) \right) dt = 0 \tag{21}$$

$$\delta \int_{t_1}^{t_2} \left(\sum_i \dot{Q}_i P_i - H(Q,P,t) \right) dt \; = \; 0 \tag{22}$$

The simultaneous validity of equations 21 and 22 does not mean that the integrands in both expressions are equal. Since the general form of the modified Hamilton's principle has zero variation at the end points, the equations 21 and 22 will be satisfied if the integrands are connected by a relation of the form

$$\lambda \left[\dot{q}_i p_i - H(q,p,t)\right] \; = \; \dot{Q}_i P_i - H(Q,P,t) + \frac{dF}{dt} \tag{23}$$

Here F is any function of the phase space coordinates with continuous second derivatives called *generating function*, and λ is a constant known as a *scale transformation*. For canonical transformationc $\lambda = 1$ and the transformation for which $\lambda \neq 1$ is called *extended canonical transformation*.

The term $\delta F/\delta t$ in equation 23 contributes to the variation of the action integral only at the end points and will therefore vanish if F is a function of (q,p,t) or (Q,P,t) or any mixture of the phase space coordinates. F is useful for specifying the exact form of the canonical transformation only when half of the variables are from the old set and half are from the new set.

If $F = F_1(q, Q, t)$, the equation 23 (with $\lambda = 1$) becomes

$$\begin{aligned} \dot{q}_i p_i - H(q,p,t) &= \dot{Q}_i P_i - H(Q,P,t) + \frac{\delta F_1}{\delta t} + \frac{\delta F_1}{\delta q_i}\dot{q}_i + \frac{\delta F_1}{\delta Q_i}\dot{Q}_i \\ H(Q,P,t) &= H(q,p,t) + \frac{\delta F_1}{\delta t} + \left(\frac{\delta F_1}{\delta q_i} - p_i\right)\dot{q}_i + \left(\frac{\delta F_1}{\delta Q_i} + P_i\right)\dot{Q}_i \qquad (24) \end{aligned}$$

Since the old and the new coordinates, q_i, and Q_i, are separately independent, equation 24 can hold identically only if the coefficients of $\dot{q}_i$, and $\dot{Q}_i$ each vanish. Thus

$$p_i = \frac{\delta F_1}{\delta q_i} \qquad (25)$$

$$P_i = -\frac{\delta F_1}{\delta Q_i} \qquad (26)$$

Equations 25 and 26 in equation 24,

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_1}{\delta t} \qquad (27)$$

The function $F(q, Q, t)$ is the generating function of the cananical transformation and it specifies the required equations of the transformation.

If $F = F_2(q, P, t) - Q_i P_i$, the equation 23 (with $\lambda = 1$) becomes

$$\dot{q}_i p_i - H(q,p,t) = \dot{Q}_i P_i - H(Q,P,t) + \frac{\delta F_2}{\delta t} + \frac{\delta F_2}{\delta q_i}\dot{q}_i + \frac{\delta F_2}{\delta P_i}\dot{P}_i - Q_i\dot{P}_i - \dot{Q}_i P_i$$

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_2}{\delta t} + \left(\frac{\delta F_2}{\delta q_i} - p_i\right)\dot{q}_i + \left(\frac{\delta F_2}{\delta P_i} - Q_i\right)\dot{P}_i \quad (28)$$

Since the old and the new coordinates, q_i, and Q_i, are separately independent, equation 24 can hold identically only if the coefficients of $\dot{q}_i$, and $\dot{Q}_i$ each vanish. Thus

$$p_i = \frac{\delta F_2}{\delta q_i} \quad (29)$$

$$Q_i = \frac{\delta F_2}{\delta P_i} \quad (30)$$

Equations 29 and 30 in equation 28,

$$H(Q,P,t) = H(q,p,t) + \frac{\delta F_2}{\delta t} \quad (31)$$

2.1. Other forms of the Generating Function

We made a particular choice to make F a function of q_i and Q_i. Given the symmetry between coordinates and canonical momenta, it is likely that we could equally well write F as a function of (q_i, P_i), (p_i, P_i) or (p_i, Q_i). These different generating functions are simply different ways to generate the same canonical transformation $(q_i, p_i) \rightarrow (Q_i, P_i)$. The four basic cananical transfornmations are given in the table 2.1.

Generating function	Generating function derivatives	
$F = F_1(q, Q, t)$	$p_i = \frac{\delta F_1}{\delta q_i}$	$P_i = -\frac{\delta F_1}{\delta Q_i}$
$F = F_2(q, P, t) - Q_i P_i$	$p_i = \frac{\delta F_2}{\delta q_i}$	$Q_i = \frac{\delta F_2}{\delta P_i}$
$F = F_3(p, Q, t) + q_i p_i$	$q_i = -\frac{\delta F_3}{\delta p_i}$	$Q_i = -\frac{\delta F_3}{\delta P_i}$
$F = F_4(p, P, t) + q_i p_i - Q_i P_i$	$p_i = -\frac{\delta F_4}{\delta p_i}$	$Q_i = \frac{\delta F_4}{\delta P_i}$

Table 1: Different forms of the Generating Function and their derivatives

Consider old and new coordinates,

$$q_i = q_i(Q, P)$$
$$p_i = p_i(Q, P)$$

$$P_i = P_i(q,p)$$

$$Q_i = Q_i(q,p)$$

$$\begin{aligned}
\frac{dQ_i}{dt} &= \frac{\delta Q_i}{\delta q_j}\frac{\delta q_j}{\delta t} + \frac{\delta Q_i}{\delta p_j}\frac{\delta p_j}{\delta t} \\
\frac{\delta H}{\delta P_i} = \dot{Q}_i &= \frac{\delta Q_i}{\delta q_j}\dot{q}_j + \frac{\delta Q_i}{\delta p_j}\dot{p}_j
\end{aligned}$$

Since,

$$\begin{aligned}
H &= H(Q,P) \\
\frac{\delta H}{\delta P_i} &= \frac{\delta H}{\delta p_j}\frac{\delta p_j}{\delta P_i} + \frac{\delta H}{\delta q_j}\frac{\delta q_j}{\delta P_i} \\
&= \dot{q}_j\frac{\delta p_j}{\delta P_i} - \dot{p}_j\frac{\delta q_j}{\delta P_i}
\end{aligned}$$

Therefore,

$$\dot{q}_j\frac{\delta p_j}{\delta P_i} - \dot{p}_j\frac{\delta q_j}{\delta P_i} = \frac{\delta Q_i}{\delta q_j}\dot{q}_j + \frac{\delta Q_i}{\delta p_j}\dot{p}_j$$

That is, the transformation is canonical only if,

$$\left(\frac{\delta p_j}{\delta P_i}\right)_{Q,P} = \left(\frac{\delta Q_i}{\delta q_j}\right)_{q,p}, \qquad \left(\frac{\delta q_i}{\delta P_i}\right)_{Q,P} = \left(\frac{\delta Q_i}{\delta p_j}\right)_{q,p} \tag{32}$$

2.2. Examples of canonical transformation

2.2.1. Generating function of the second type

If $F_2 = q_i P_i$

$$\begin{aligned} \frac{\delta F_2}{\delta q_i} &= P_i \\ \frac{\delta F_2}{\delta P_i} &= q_i \end{aligned} \tag{33}$$

On comparing the values with the table 2.1.,

$$\begin{aligned} \frac{\delta F_2}{\delta q_i} &= P_i = p_i \\ \frac{\delta F_2}{\delta P_i} &= q_i = Q_i \end{aligned} \tag{34}$$

Hence $H(q,p,t) = H(Q,P,t)$ and F_2 generates the identity transformation.

2.2.2. Generating function of the first type

If $F_1 = q_i Q_i$

$$\begin{aligned}\frac{\delta F_1}{\delta q_i} &= Q_i \\ \frac{\delta F_1}{\delta Q_i} &= q_i\end{aligned} \tag{35}$$

On comparing the values with the table 2.1.,

$$\begin{aligned}\frac{\delta F_1}{\delta q_i} &= Q_i = p_i \\ \frac{\delta F_1}{\delta Q_i} &= q_i = -P_i\end{aligned} \tag{36}$$

Thus the transformation interchanges the momenta and the coordinates.

2.2.3. Simple harmonic oscillator

The Hamiltonian for a simple harmonic oscillator can be written as

$$H(q_i, p_i) = \frac{1}{2m}\left(\sum_i p_i^2 + m^2\omega^2 \sum_i q_i^2\right) \tag{37}$$

This form of the Hamiltonian, as the sum of two squares, suggests a transformation in which $H(q, p, t)$ is cyclic in the new coordinate. Then a canonical transformation takes the form

$$p_i = f(P)\, cos\, Q_i \tag{38}$$

$$q_i = \frac{f(P)}{m\omega}\, sin\, Q_i \tag{39}$$

Substituting for p_i^2 and q_i^2 by using equations 38 and 39 to equation 37,

$$H(Q_i, P_i) = \frac{1}{2m}\left(\sum_i f(P)^2 cos^2\, Q_i + m^2\omega^2 \sum_i \frac{f(P)^2}{m^2\omega^2} sin^2\, Q_i\right)$$

$$H(Q_i, P_i) = \frac{f(P)^2}{2m} \tag{40}$$

If we use a generating function given by

$$F_1 = \frac{m\omega q_i^2}{2} cot\, Q_i \tag{41}$$

Then

$$p_i = \frac{\delta F_1}{\delta q_i} = m\omega q_i \, cot\, Q_i$$

$$P_i = -\frac{\delta F_1}{\delta Q_i} = \frac{m\omega q_i^2}{2 sin^2 Q_i}$$

$$q_i = \sqrt{\frac{2P_i}{m\omega}} sin\, Q_i \tag{42}$$

$$p_i = \sqrt{2P_i m\omega}\, cos\, Q_i \tag{43}$$

On comparing equations 42 and 43 with equations 38 and 39,

$$f(P_i) = \sqrt{2P_i m\omega} \tag{44}$$

Equation 44 in equation 40 gives

$$H = \omega \sum_i P_i = \sum_i E_i \tag{45}$$

$$\sum_i P_i = \frac{\sum_i E_i}{\omega}$$

where E_i is the total energy of a oscillator. The equations of motion in the cananical coordinates is

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} = \omega \quad (46)$$
$$Q_i = \omega t + \alpha \quad (47)$$

where α is the constant of integration evaluated by the initial conditions. From equations 42 and 43, the solutions for p and q are written as

$$q_i = \sqrt{\frac{2E_i}{m\omega^2}}\, sin\,(\omega t + \alpha) \quad (48)$$
$$p_i = \sqrt{2mE_i}\, cos\,(\omega t + \alpha) \quad (49)$$

From equation 42 and 43, it can see that we have transformed from simple position q_i and momentum p_i to phase Q_i and energy P_i of the oscillatory motion. Equation 48 shows that the energy depends only on the oscillator amplitude. This kind of transformation is going to have obvious use when dealing with mechanical or electromagnetic waves.

3. Poisson brackets

Poisson brackets are a powerful and sophisticated tool in the Hamiltonian formalism of Classical Mechanics. They also happen to provide a direct link between classical and quantum mechanics. A classical system with N degrees of freedom, say a set of $N/3$ particles in three dimensions, is described by $2N$ phase space coordinates. These are the N generalized coordinates $q_1, q_2, q_3, ..., q_N$ and N conjugate momenta $p_1, p_2, p_3, ..., p_N$. The Hamiltonian of the system depends on these $2N$ variables and possibly on time t as well, and it can be expressed as

$$H(q_1, q_2, q_3, ..., q_N, p_1, p_2, p_3, ..., p_N, t)H(q_i, p_i, t)$$

The Poisson bracket is an operation which takes two functions of phase space and time, call them $F(q_i, p_i, t)$ and $G(q_i, p_i, t)$ and produces a new function. With respect to cananical coordinates (q_i, p_i), it is defined as

$$[F, G] = \sum_{i}^{N} \left(\frac{\delta F}{\delta q_i} \frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i} \frac{\delta G}{\delta q_i} \right) \tag{50}$$

In the case of a single degree of freedom, $N = 1$, phase space is 2-dimensional, (q, p) and the Poisson bracket has only two terms

$$[F, G] = \left(\frac{\delta F}{\delta q} \frac{\delta G}{\delta p} - \frac{\delta F}{\delta p} \frac{\delta G}{\delta q} \right) \tag{51}$$

Time derivative of the function $F(q_i, p_i, t)$ is

$$\begin{aligned} \frac{dF}{dt} &= \sum_i^N \left(\frac{\delta F}{\delta q_i} \frac{\delta q_i}{\delta t} + \frac{\delta F}{\delta p_i} \frac{\delta p_i}{\delta t} \right) + \frac{\delta F}{\delta t} \\ \frac{dF}{dt} &= \sum_i^N \left(\frac{\delta F}{\delta q_i} \dot{q}_i + \frac{\delta F}{\delta p_i} \dot{p}_i \right) + \frac{\delta F}{\delta t} \end{aligned} \tag{52}$$

By using the Hamiltonian equations of motion

$$\dot{q}_i = \frac{\delta H}{\delta p_i} \qquad \dot{p}_i = -\frac{\delta H}{\delta q_i}$$

equation 52 becomes,

$$\frac{dF}{dt} = \sum_i^N \left(\frac{\delta F}{\delta q_i} \frac{\delta H}{\delta p_i} - \frac{\delta F}{\delta p_i} \frac{\delta H}{\delta q_i} \right) + \frac{\delta F}{\delta t} \tag{53}$$

$$\frac{dF}{dt} = [F, H] + \frac{\delta F}{\delta t} \tag{54}$$

Equation 53 is the equation of motion of the function F expressed in terms of Poisson bracket.

3.1. Properties of Poisson bracket

1. Consider

$$\begin{aligned}[F, G_1 + G_2] &= [F, G_1] + [F, G_2] \\ [F, G_1 G_2] &= [F, G_1]G_2 + G_1[F, G_2]\end{aligned}$$

2. The Poisson bracket is anti-symmetric in its two arguments

$$[G, F] = -[F, G] \tag{55}$$

   An immediate consequence of this is that $[F, F] = 0$ for any function at all.

$$\begin{aligned}[q_j, q_k] &= 0 = [p_j, p_k] \\ [q_j, p_k] &= \delta_{jk} = [p_j, q_k]\end{aligned}$$

3. The Poisson bracket is linear in either of its arguments

$$\begin{aligned} [F_1 + F_2, G] &= [F_1, G] + [F_2, G] \\ [F, G_1 + G_2] &= [F, G_1] + [F, G_2] \\ [F, G_1 G_2] &= [F, G_1]G_2 + G_1[F, G_2] \end{aligned}$$

4. **Invariance under canonical transformations:**

$$[F, G]_{(q,p)} = \sum_i^N \left(\frac{\delta F}{\delta q_i}\frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i}\frac{\delta G}{\delta q_i} \right)$$

$$= \left(\frac{\delta F}{\delta q_1}\frac{\delta G}{\delta p_1} - \frac{\delta F}{\delta p_1}\frac{\delta G}{\delta q_1} \right) + \left(\frac{\delta F}{\delta q_2}\frac{\delta G}{\delta p_2} - \frac{\delta F}{\delta p_2}\frac{\delta G}{\delta q_2} \right) + \ldots\ldots\ldots\ldots$$

$$+ \left(\frac{\delta F}{\delta q_n}\frac{\delta G}{\delta p_n} - \frac{\delta F}{\delta p_n}\frac{\delta G}{\delta q_n} \right)$$

$$= \left[\frac{\delta F}{\delta q_1} \quad \frac{\delta F}{\delta q_2} \ldots\ldots \quad \frac{\delta F}{\delta q_n} \quad \frac{\delta F}{\delta p_1} \quad \frac{\delta F}{\delta p_2} \ldots\ldots \quad \frac{\delta F}{\delta p_n} \right] \begin{bmatrix} 0 & 0\ldots\ldots & 1 & 0 \\ 0 & 0\ldots\ldots & 0 & 1 \\ . & & & \\ . & & & \\ . & & & \\ -1 & 0\ldots\ldots & 0 & 0 \\ 0 & -1\ldots\ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta G}{\delta q_1} \\ \frac{\delta G}{\delta q_2} \\ . \\ . \\ \frac{\delta G}{\delta q_n} \\ \frac{\delta G}{\delta p_1} \\ \frac{\delta G}{\delta p_2} \\ . \\ . \\ \frac{\delta G}{\delta p_n} \end{bmatrix}$$

$$[F, G]_\eta = \widetilde{\left[\frac{\delta F}{\delta \eta}\right]} \mathbf{J} \left[\frac{\delta G}{\delta \eta}\right]$$

where η consists of $2n$ elements of (q_i, p_i) and $\mathbf{J}$ is the $2n \times 2n$ anti-symmetric matrix.

For $i = 2$, the above equation can be written as

$$[F,G]_{(q,p)} = \begin{bmatrix} \frac{\delta F}{\delta q_1} & \frac{\delta F}{\delta q_2} & \frac{\delta F}{\delta p_1} & \frac{\delta F}{\delta p_2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta G}{\delta q_1} \\ \frac{\delta G}{\delta q_2} \\ \frac{\delta G}{\delta p_1} \\ \frac{\delta G}{\delta p_2} \end{bmatrix}$$

η are the old coordinates (q_i, p_i) and $\xi = \xi(\eta)$ are the transformed coordinates (Q_i, P_i), then

$$\begin{aligned} \frac{\delta F}{\delta \eta_{\mathbf{i}}} &= \frac{\delta F}{\delta \xi_j}\frac{\delta \xi_j}{\delta \eta_i} \\ &= \frac{\delta F}{\delta \xi_1}\frac{\delta \xi_1}{\delta \eta_i} + \frac{\delta F}{\delta \xi_2}\frac{\delta \xi_2}{\delta \eta_i} + \ldots\ldots + \frac{\delta F}{\delta \xi_n}\frac{\delta \xi_n}{\delta \eta_i} \\ &= \begin{bmatrix} \frac{\delta \xi_1}{\delta \eta_i} & \frac{\delta \xi_2}{\delta \eta_i} \ldots\ldots \frac{\delta \xi_n}{\delta \eta_i} \end{bmatrix} \begin{bmatrix} \frac{\delta F}{\delta \xi_1} \\ \frac{\delta F}{\delta \xi_2} \\ . \\ . \\ \frac{\delta F}{\delta \xi_n} \end{bmatrix} \end{aligned}$$

$$\left[\frac{\delta F}{\delta \eta}\right] = \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} \left[\frac{\delta F}{\delta \xi}\right]$$

Similarly,

$$\left[\frac{\delta G}{\delta \eta}\right] = \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} \left[\frac{\delta G}{\delta \xi}\right]$$

Therefore,

$$\begin{aligned} [F,G]_\eta &= \widetilde{\widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} \left[\frac{\delta F}{\delta \xi}\right]} \mathbf{J} \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} \left[\frac{\delta G}{\delta \xi}\right] \\ &= \widetilde{\left[\frac{\delta F}{\delta \xi}\right]} \left[\frac{\delta \xi}{\delta \eta}\right] \mathbf{J} \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} \left[\frac{\delta G}{\delta \xi}\right] \end{aligned}$$

If the transformation $\eta \longrightarrow \xi$ is canonical,

$$\left[\frac{\delta \xi}{\delta \eta}\right] \mathbf{J} \widetilde{\left[\frac{\delta \xi}{\delta \eta}\right]} = \mathbf{J}$$

Therefore,

$$\begin{aligned} [F,G]_\eta &= \left[\frac{\delta \xi}{\delta \eta}\right] \mathbf{J} \left[\frac{\delta G}{\delta \xi}\right] \\ &= [F,G]_\xi \\ [F,G]_{(q,p)} &= [F,G]_{(Q,P)} \end{aligned}$$

3.2. Constants of the Motion

A constant of the motion is some function of phase space, independent of time, $F(q_i, p_i)$, whose value is constant for any particle. In other words, $F(q_i, p_i)$ is a constant of the motion if

$$\frac{dF}{dt} = 0.$$

Since we specified that F does not depend explicitly in time it follows that

$$\frac{\delta F}{\delta t} = 0$$

Then from equation 54,

$$[F, H] = 0$$

Thus F is a constant of the motion if and only if $[F, H] = 0$ for all points in phase space.

- **Energy:** Due to the anti-symmetry of the Poisson bracket $[H, H] = 0$. Using this in equation 54,

$$\frac{dH}{dt} = \frac{\delta H}{\delta t} \tag{56}$$

If the Hamiltonian does not depend on time explicitly,

$$\frac{\delta H}{\delta t} = 0$$

Then

$$\frac{dH}{dt} = 0, \qquad H(q_i, p_i) = constant \tag{57}$$

That is *energy is conserved in cases where the Hamiltonian is time-independent.*

- **Linear Momentum:** In a case where the Hamiltonian does not contain a particular coordinate, q_i , explicitly it is said to be cyclic in that coordinate. Then

$$[p_i, H] = \frac{\delta p_i}{\delta q_i}\frac{\delta H}{\delta p_i} - \frac{\delta p_i}{\delta p_i}\frac{\delta H}{\delta q_i} = -\frac{\delta H}{\delta q_i} \qquad \left(\frac{\delta p_i}{\delta q_i} = 0\right) \tag{58}$$

Since q_i is cyclic, $(\delta H/\delta q_i) = 0$, then $[p_i, H] = 0$, so p_k is a constant of the motion. Thus the *momentum is conserved if it is conjugate to a cyclic coordinate.*

- **Angular Momentum:** Consider a particle in three dimension, (x, y, z), subject to a central force potential $V(r) = V(x, y, z)$. The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) \\ \frac{\delta T}{\delta p_x} &= \frac{p_x}{m}, \qquad \frac{\delta T}{\delta p_y} = \frac{p_y}{m}, \qquad \frac{\delta T}{\delta p_z} = \frac{p_z}{m} \end{aligned} \tag{59}$$

The potential energy of the system is

$$V = V(r) \;=\; V\left(\sqrt{x^2 + y^2 + z^2}\right)$$

$$\frac{\delta V}{\delta x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} V'(r), \quad \frac{\delta V}{\delta y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} V'(r), \quad \frac{\delta V}{\delta z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} V'(r) \tag{60}$$

where $V'(r)$ is the potential function. The Hamiltonian of the system is

$$H \;=\; \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) \tag{61}$$

Angular momentum of the system is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

and the components of L can be written as

$$\begin{aligned} L_z &= xp_y - yp_x \\ L_y &= zp_x - xp_z \\ L_x &= yp_z - zp_x \end{aligned}$$

Consider

$$\begin{aligned} [L_z, H] &= [L_z, T + V] \\ [L_z, H] &= [L_z, T] + [L_z, V] \end{aligned}$$

$$\begin{aligned} [L_z, T] &= \frac{\delta L_z}{\delta q_i}\frac{\delta T}{\delta p_i} - \frac{\delta L_z}{\delta p_i}\frac{\delta T}{\delta q_i} \\ &= \frac{\delta L_z}{\delta x}\frac{\delta T}{\delta p_x} - \frac{\delta L_z}{\delta p_x}\frac{\delta T}{\delta x} + \frac{\delta L_z}{\delta y}\frac{\delta T}{\delta p_y} - \frac{\delta L_z}{\delta p_y}\frac{\delta T}{\delta y} \\ &= \frac{\delta L_z}{\delta x}\frac{\delta T}{\delta p_x} + \frac{\delta L_z}{\delta y}\frac{\delta T}{\delta p_y} \qquad \left(\frac{\delta T}{\delta q_i} = 0 \right) \\ &= \frac{\delta}{\delta x}(xp_y - yp_x)\,\frac{\delta T}{\delta p_x} + \frac{\delta}{\delta y}(xp_y - yp_x)\frac{\delta T}{\delta p_y} \end{aligned} \tag{62}$$

By using equation 59,

$$[L_z, T] = p_y \frac{p_x}{m} - p_x \frac{p_y}{m} = 0 \quad (63)$$

$$\begin{aligned}
[L_z, V] &= \frac{\delta L_z}{\delta q_i}\frac{\delta V}{\delta p_i} - \frac{\delta L_z}{\delta p_i}\frac{\delta V}{\delta q_i} \\
&= \frac{\delta L_z}{\delta x}\frac{\delta V}{\delta p_x} - \frac{\delta L_z}{\delta p_x}\frac{\delta V}{\delta x} + \frac{\delta L_z}{\delta y}\frac{\delta V}{\delta p_y} - \frac{\delta L_z}{\delta p_y}\frac{\delta V}{\delta y} \\
&= -\frac{\delta L_z}{\delta p_x}\frac{\delta V}{\delta x} - \frac{\delta L_z}{\delta p_y}\frac{\delta V}{\delta y} \qquad \left(\frac{\delta V}{\delta p_i} = 0 \right) \\
&= -\frac{\delta}{\delta p_x}(xp_y - yp_x)\,\frac{\delta V}{\delta x} - \frac{\delta}{\delta p_y}(xp_y - yp_x)\frac{\delta V}{\delta y} \\
[L_z, V] &= y\frac{\delta V}{\delta x} - x\frac{\delta V}{\delta y}
\end{aligned}$$

By using equation 60,

$$[L_z, V] = y\frac{x}{\sqrt{x^2+y^2+z^2}}V'(r) - x\frac{y}{\sqrt{x^2+y^2+z^2}}V'(r) = 0 \quad (64)$$

Equations 63 and 63 gives

$$[L_z, H] = [L_z, T] + [L_z, V] = 0$$

Similarly we can show that $[L_x, H] = 0$, $[L_y, H] = 0$. Therefore for *a particle moving in a central force potential all three components of angular momentum are conserved.*

3.3. Angular momentum and Poisson bracket relations

Angular momentum of the system is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$

The components of L in Cartesian coordinates are

$$\begin{aligned} L_z &= xp_y - yp_x \\ L_y &= zp_x - xp_z \\ L_x &= yp_z - zp_x \end{aligned}$$

If F is a vector rotating about z axis, the equation of motion in terms of Poisson bracket is

$$\frac{dF}{d\theta} = [F, L_z] \tag{65}$$

Also we can write

$$\frac{d\vec{F}}{d\theta} = \hat{k} \times \vec{F} \tag{66}$$

By using equations 65 and 66,

$$[F, L_z] = \hat{k} \times \vec{F} \tag{67}$$

If $\vec{F} = \vec{r} = (ix + jy + kz)$, then

$$\begin{aligned} [F, L_z]_x = [x, L_z] &= \hat{k} \times (ix) \\ [x, L_z] &= -y \\ \text{Similarly,} \quad [\mathrm{F}, \mathrm{L_z}]_\mathrm{y} = [\mathrm{y}, \mathrm{L_z}] &= x \\ [F, L_z]_z = [z, L_z] &= 0 \end{aligned} \tag{68}$$

Then we can write general relation as

$$[q_i, L_j] = \epsilon_{ijk} q_k$$

where $\epsilon_{ijk} = 0$ for $i = j$ or $j = k$, $\epsilon_{ijk} = 1$ for ijk are distinct and in cyclic order and $\epsilon_{ijk} = -1$ for ijk are distinct and not in cyclic order.

If $\vec{F} = \vec{P} = (ip_x + jp_y + kp_z)$, then

$$\begin{aligned} [F, L_z]_x = [p_x, L_z] &= \hat{k} \times (ip_x) \\ [p_x, L_z] &= -p_y \\ Similarly, \quad [F, L_z]_y = [p_y, L_z] &= p_x \\ [F, L_z]_z = [p_z, L_z] &= 0 \end{aligned}$$

The general relation is

$$[p_i, L_j] = \epsilon_{ijk} p_k$$

If $\vec{F} = \vec{L} = (iL_x + jL_y + kL_z)$, then

$$\begin{aligned} [F, L_z]_x = [L_x, L_z] &= \hat{k} \times (iL_x) \\ [L_x, L_z] &= -L_y \\ Similarly, \quad [F, L_z]_y = [L_y, L_z] &= L_x \end{aligned}$$

$$[F, L_z]_z = [L_z, L_z] \ = \ 0 \tag{69}$$

Again consider

$$\begin{aligned}
[L_x, L_y] &= [yp_z - zp_x, L_y] \\
&= [yp_z, L_y] - [zp_x, L_y] \\
&= [yp_z, zp_x - xp_z] - [zp_x, zp_x - xp_z] \\
&= [yp_z, zp_x] - [yp_z, xp_z] - [zp_x, zp_x] + [zp_x, xp_z] \\
&= [yp_z, zp_x] + [zp_y, xp_z] \qquad ([yp_z, xp_z] = 0, \quad [zp_x, zp_x] = 0) \\
&= xp_y - yp_x \\
[L_x, L_y] &= L_z, \qquad [L_y, L_x] = -L_z \\
\textit{Similarly,} \quad [L_y, L_z] &= L_x, \qquad [L_z, L_y] = -L_x \\
[L_z, L_x] &= L_y, \qquad [L_x, L_z] = -L_y
\end{aligned}$$

Thus the general relation can be written as

$$[L_i, L_j] \ = \ \epsilon_{ijk} L_k$$

where $\epsilon_{ijk} = 0$ for $i = j$ or $j = k$, $\epsilon_{ijk} = 1$ for ijk are distinct and in cyclic order and $\epsilon_{ijk} = -1$ for ijk are distinct and not in cyclic order.

4. Hamilton-Jacobi equation

The Hamilton-Jacobi equation makes use of a special canonical transformation to convert the standard Hamiltonian problem of $2N$ first-order ordinary differential equations in $2N$ variables into a single first-order partial differential equation with $N+1$ partial derivatives with respect to the q_i and time.

If a canonical transformation from some arbitrary set of generalized coordinates (q, p) to some new set (Q, P) such that all the Q and P are constant in time, then

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} = 0 \qquad \dot{P}_i = -\frac{\delta H}{\delta Q_i} = 0 \tag{70}$$

One way to guarantee the above conditions is to require that

$$H(Q, P) = 0$$

Equation for the transformation of the Hamiltonian under a canonical transformation, (equation 31), becomes

$$H(q, p, t) + \frac{\delta F}{\delta t} = 0 \tag{71}$$

Since the new momenta will be constant, it is sensible to make F a function of the type F_2, $F = S(q, P)$. Then

$$p_i = \frac{\delta S}{\delta q_i} \tag{72}$$

The equation 71 becomes,

$$H\left(q, \frac{\delta S}{\delta q_i}, t\right) + \frac{\delta S}{\delta t} \ = \ 0 \tag{73}$$

Equation 73 is called Hamilton-Jacobi equation, constitutes a partial differential equation of N independent cocordinates $q_1, q_2, q_3, ..., q_N$ and t. That is there are $N + 1$ variables (N initial values of q and a constant energy). S is known as Hamilton's principle function.

Since a solution S of the equation 73 will generate a transformation that makes the N components of P constant, and since S is a function of the P, the P can be taken to be the N constants. Independent of the above equation, we know that there must be N additional constants to specify the full motion. These are the Q. The existence of these extra constants is not implied by the Hamilton-Jacobi equation, since it only needs $N + 1$ constants to find a full solution S. The additional N constants exist because of Hamilton's equations, which require $2N$ initial conditions for a full solution.

Since the P and Q are constants, it is conventional to refer to them with the symbols $\alpha_i = P_i$ and $\beta_i = Q_i$. The full solution $q(t), p(t)$ to the problem is found by making use of the generating function S and the initial conditions $q(0)$ and $p(0)$. Then the function S is

$$S = S(q, P, t) = S(q_1, q_2, ..., q_N, \alpha_1, \alpha_2, \alpha_3, ...\alpha_{N+1}, t) = S(q, \alpha, t)$$

where quantities $\alpha_1, \alpha_2, \alpha_3, ...\alpha_{N+1}$ are $N + 1$ independent constants of integration. The generating function partial derivative relations are

$$p_i = \frac{\delta S}{\delta q_i} \qquad \beta_i = \frac{\delta S}{\delta \alpha_i} = Q_i \tag{74}$$

The constants α and β are found by applying the above equations at $t = 0$.

The time derivative of S can be written as

$$\frac{dS}{dt} = \sum_i \frac{\delta S}{\delta q_i}\frac{dq_i}{dt} + \frac{\delta S}{\delta t} \tag{75}$$

By using equations 72 and 73 in equation 75

$$\frac{dS}{dt} = \sum_i p_i \dot{q}_i - H = L \tag{76}$$

$$S = \int L\, dt + constant \tag{77}$$

This is an interesting result - that the action integral is the generator of the canonical transformation that corresponds to time evolution.

When the Hamiltonian does not depend explicitly upon the time, Hamilton's principle function S can be written in the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \tag{78}$$

$$\frac{\delta S}{\delta q_i} = \frac{\delta W}{\delta q_i}, \qquad \frac{\delta S}{\delta t} = -\alpha = -E \tag{79}$$

where $\alpha = E$ is the time independent value of H and $W(q, \alpha)$ is called *Hamilton's characteristic function.*

The time derivative of $W(q, \alpha)$ is

$$\frac{dW}{dt} = \frac{\delta W}{\delta q_i} \dot{q}_i \tag{80}$$

By using equations 74 and 79 in equation 80,

$$\frac{dW}{dt} = p_i \dot{q}_i \tag{81}$$

$$W = \int p_i \dot{q}_i dt = \int p_i dq_i \tag{82}$$

which is known as the *abbreviated action.*

4.1. Simple harmonic oscillator

The simple harmonic oscillator Hamiltonian is

$$H = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right) = E \tag{83}$$

where E is the time independent value of H. The Hamilton-Jacobi equation for this Hamiltonian can be written by using equation 74 as

$$\frac{1}{2m}\left[\left(\frac{\delta S}{\delta q}\right)^2 + m^2\omega^2 q^2\right] + \frac{\delta S}{\delta t} = 0 \tag{84}$$

Since H is conserved, $\frac{\delta S}{\delta t} = constant = -\alpha$ and by using equation 79

$$\frac{1}{2m}\left[\left(\frac{\delta W}{\delta q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

$$\frac{\delta W}{\delta q} = \sqrt{2m\alpha - m^2\omega^2 q^2} \tag{85}$$

$$W = \sqrt{2m\alpha}\int\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}}\,dq \tag{86}$$

Since $S = W - \alpha t$, then

$$S = \sqrt{2m\alpha}\int\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}}dq - \alpha t \tag{87}$$

$$\frac{\delta S}{\delta \alpha} = \sqrt{\frac{m}{2\alpha}}\int\frac{dq}{\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}}} - t$$

$$\beta = \frac{1}{\omega}\int\frac{\sqrt{\frac{m\omega^2}{2\alpha}}}{\sqrt{1-\frac{m\omega^2 q^2}{2\alpha}}}dq - t$$

$$\beta + t = \frac{1}{\omega} arc\sin\sqrt{\frac{m\omega^2}{2\alpha}}q$$

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \omega\beta)$$

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \phi) \tag{88}$$

Again consider equation 85,

$$\frac{\delta W}{\delta q} = \sqrt{2m\alpha\left(1-\frac{m\omega^2q^2}{2\alpha}\right)}$$
$$p = \sqrt{2m\alpha[1-\sin^2(\omega t+\phi)]}$$
$$p = \sqrt{2m\alpha}\,\cos(\omega t+\phi) \tag{89}$$

At time $t=0$, the equations 88 and 89 becomes

$$q_o = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\,\phi \tag{90}$$
$$p_o = \sqrt{2m\alpha}\cos\,\phi \tag{91}$$

On squaring and adding the above equations, we get

$$2m\alpha = p_o^2m^2\omega^2q_o^2 \tag{92}$$

Thus α can be obtained in terms of p_o and q_o. Equations 90/91 gives

$$\tan\,\phi = m\omega\frac{q_o}{p_o} \tag{93}$$

When $q_o=0$, $\beta=0$ corresponds to starting the motion with the oscillator at its equilibrium position $q=0$.

Thus Hamilton's principle function is the generator of a canonical transformation to a new coordinate that measures the phase angle of the oscillation and to a new canonical momentum identified as the total energy.

By using equation 86, the Hamilton's principle function can be written as

$$S = \sqrt{2m\alpha}\int\sqrt{1-\frac{m\omega^2q^2}{2\alpha}}\,dq - \alpha t$$

On substituting for q and dq by using equation 88,

$$\begin{aligned} S &= 2\alpha\int\cos^2(\omega t+\phi)dt - \alpha t \\ &= \alpha\int\left(2\cos^2(\omega t+\phi)-1\right)dt \\ S &= \alpha\int\left(\cos^2(\omega t+\phi)-\sin^2(\omega t+\phi)\right)dt \qquad (94) \end{aligned}$$

from equations 88 and 89, we can get

$$S = \int\left(\frac{p^2}{2m}-\frac{m\omega^2q}{2}\right)\,dt \qquad (95)$$

$$S = \int L\,dt \qquad (96)$$

i.e., S is the time integral of the Lagrangian.