Classical Mechanics

Dr. M Ramegowda

Dept. of Physics

Govt. College (Autonomous), Mandya

1. Rigid body dynamics

A rigid body is defined as a system of mass points subject to the holonomic constraints that the distances between all pairs of points remain constant throughout the motion.

1.1. The independent coordinates of a rigid body - Degrees of freedom

A rigid body with N particles can at most have $3N$ degrees of freedom, but these are greatly reduced by the constraints, which can be expressed as equations of the form

$$r_{ij} = c_{ij} \tag{1}$$

where r_{ij} is distance between any i^{th} and j^{th} particle and c_{ij} is constant. But all these relations are not independent. Consider three non collinear points *1, 2* and *3* of rigid body, as shown in the figure 1. If there is no constraint equations, the number of degrees of freedom becomes 9. If we write the constraint equations,

$$r_{12} = c_{12}, \qquad r_{23} = c_{23} \qquad r_{31} = c_{31} \tag{2}$$

Then the number of degrees of freedom becomes $9 - 3 = 6$. That is only six coordinates are needed. Of these 6 coordinates, 3 are the coordinates of CM of rigid body with respect to the fixed axis and the other three generalized coordinates are the angles that the axis of rotation makes with the fixed coordinate system. A rigid body in space thus

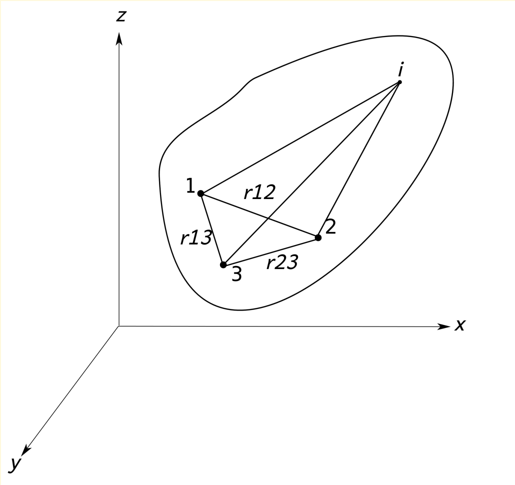

Figure 1: Rigid body with three non collinear points.

needs six independent generalized coordinates to specify its configuration, no matter how many particles it may contain.

2. Rotation about an axis, Orthogonal matrix

Consider a rigid body in which Cartesian coordinate system x', y', z' fixed at the point O'. $\vec{r'}$ is the position vector of a mass point rotated spacially at an angle θ. The coordinate system x', y', z' is also rotates with angles θ_{ij} as shown in the figure 2. Note that the angle θ_{ij} is defined so that the first index refers to the primed system and the second index to the unprimed system. Thus from the figure 2 we can write,

$$
\begin{aligned}
\vec{r'} &= \vec{r} && (3)\\
i'x' + j'y' + k'z' &= ix + jy + kz \\
(i'x' + j'y' + k'z').i' &= (ix + jy + kz).i' \\
x' &= (i.i')\ x + (j.i')\ y + (k.i')\ z \\
x' &= cos\theta_{11}\ x + cos\theta_{12}\ y + cos\theta_{13}\ z \\
Similarly \quad y' &= cos\theta_{21}\ x + cos\theta_{22}\ y + cos\theta_{23}\ z \\
z' &= cos\theta_{31}\ x + cos\theta_{32}\ y + cos\theta_{33}\ z && (4)
\end{aligned}
$$

Equations 4 constitute a group of transformation equations from a set of coordinates x, y, z to a new set x', y', z' with direction cosines $cos\theta_{ij}$ as transformation coefficients. They form an example of a *linear or vector* transformation, defined by transformation

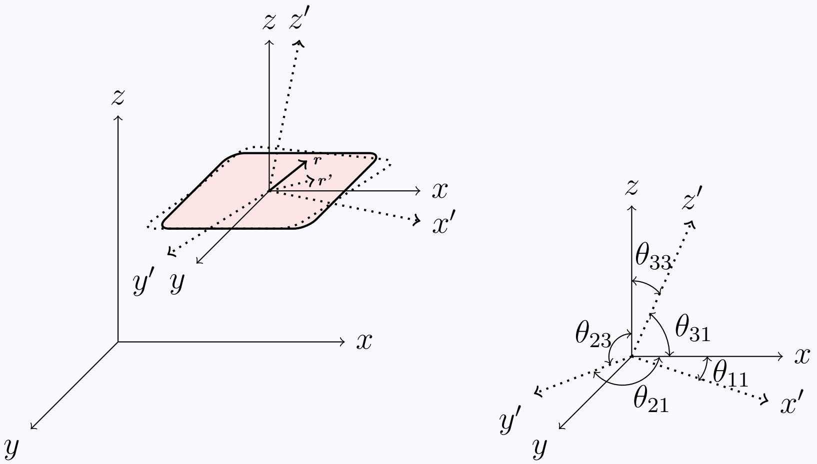

Figure 2: Rotation of a rigid body.

equations of the form

$$
\begin{aligned}
x' &= a_{11}\ x + a_{12}\ y + a_{13}\ z \\
y' &= a_{21}\ x + a_{22}\ y + a_{23}\ z \\
z' &= a_{31}\ x + a_{32}\ y + a_{33}\ z
\end{aligned}
\tag{5}
$$

where $a_{ij} = cos\theta_{ij}$. The equations 5 can be written in the form of matrix equation as

$$
\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\tag{6}
$$

$$[X'] \equiv [A]\,[X] \tag{7}$$

where $[X']$ is a column matrix with components (x', y', z'); $[X]$ is a similar column matrix with components (x, y, z) and $[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is the rotation matrix.

Consider a rigid body rotated in $x - y$ plane by an angle ϕ as shown in the figure 3. Then

$$\begin{aligned} a_{11} &= cos\,\phi & a_{12} = sin\,\phi && a_{13} = 0 \\ a_{21} &= -sin\,\phi & a_{22} = cos\,\phi && a_{23} = 0 \\ a_{31} &= 0 & a_{32} = 0 && a_{33} = 1 \end{aligned} \tag{8}$$

The rotation matrix A can be written as

$$A = \begin{bmatrix} cos\,\phi & sin\,\phi & 0 \\ -sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \tag{9}$$

$$\begin{aligned} A\tilde{A} &= \begin{bmatrix} cos\,\phi & sin\,\phi & 0 \\ -sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\,\phi & -sin\,\phi & 0 \\ sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ A\tilde{A} &= 1 \end{aligned} \tag{10}$$

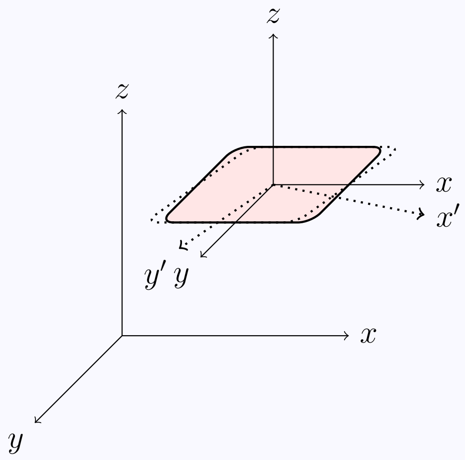

Figure 3: Rotation of a rigid body in $x - y$ plane.

From the equation 10, it is clear that $\tilde{A} = A^{-1}$. That is, the rotational matrix A is orthoganal and it can be taken as an operator. The matrix A operating on the components of a vector in the unprimed system yields the components of the vector in the primed system. Symbolically, the process can be written as,

$$\vec{r}\,' = A\vec{r} = \vec{r} \tag{11}$$

That is, the transformation matrix $[A]$ affects rotation of the rigid body with one point fixed has eigenvalue + 1. This is called Euler's theorem which states that *A general displacement of a rigid body with one point fixed is a rotation about some axis.*

A more general theorem than Euler's is proved by Chasles and it states that, *The most* *general displacement of rigid body is a translation of the rigid body plus rotation of the rigid body.*

It can be shown that, the orthogonal matrix whose determinant is -1 represents inversion and cannot represent physical displacement of a rigid body.

3. Euler angles

Eulerian angles are three rotations about three independent axes chosen in a certain successive way.

Consider a rigid body with initial system of axes xyz. Rotate the rigid body by by and angle ϕ counterclockwise about the z axis as shown in the figure 3, and the resultant coordinate system is labeled as $\xi\eta\zeta$. Then,

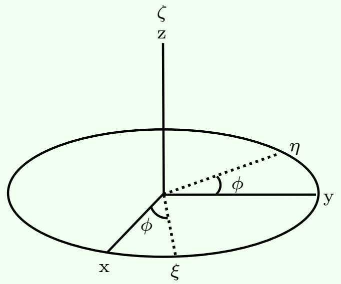

Figure 4: Rotation about z axis

$$
\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} cos\,\phi & sin\,\phi & 0 \\ -sin\,\phi & cos\,\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{12}
$$

$$
\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = [A_\phi] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{13}
$$

In the second stage, the intermediate axes, $\xi\eta\zeta$ are rotated about the ξ axis counter-

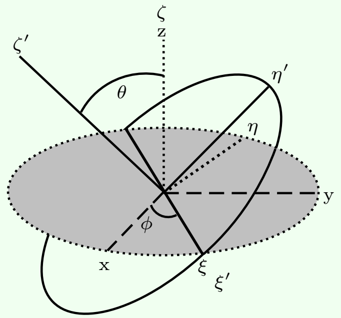

Figure 5: Rotation about ξ' axis

clockwise by an angle θ to produce another intermediate set, $\xi'\eta'\zeta'$ axes as shown in the figure 3. The ξ' axis is at the intersection of the xy and $\xi'\eta'$ planes and is known as the

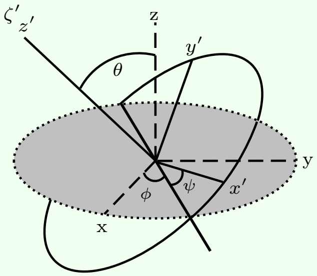

Figure 6: Rotation about ζ' axis

line of nodes. The transformation then is written as,

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\,\theta & sin\,\theta \\ 0 & -sin\,\theta & cos\,\theta \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{14}$$

$$\begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} = [A_\theta] \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \tag{15}$$

Finally, $\xi'\eta'\zeta'$ axes are rotated counterclockwise by an angle ψ about the ζ' axis as shown in the figure 6 to produce the desired $x'y'z'$ system of axes.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} cos\,\psi & sin\,\psi & 0 \\ -sin\,\psi & cos\,\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [A_\psi] \begin{bmatrix} \xi' \\ \eta' \\ \zeta' \end{bmatrix} \tag{17}$$

Then, the transformation of axes (xyz) to $(x'y'z')$ can be written by using the equations 13, 15 and 17 as,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [A_\phi]\,[A_\theta]\,[A_\psi] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [A] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{19}$$

where $[A] = [A_\phi]\,[A_\theta]\,[A_\psi]$. By using equations 12, 14 and 16,

$$
\begin{aligned}
A &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
A &= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \qquad (20)
\end{aligned}
$$

It can be shown that $\tilde{A} = A^{-1}$, hence A is orthoganal.

4. Angular momentum and kinetic energy of motion about a point

Consider a rigid body moves with one point stationary, the total angular momentum about that point is

$$\vec{L} = \sum_i m_i(\vec{r_i} \times \vec{v_i}) \tag{21}$$

where $\vec{r_i}$ and $\vec{v_i}$ are the radius vector and velocity, respectively, of the *ith* particle relative to the given point. Since $\vec{r}$, is a fixed vector relative to the body, the velocity $\vec{v}$, with respect to the fixed frame arises solely from the rotational motion of the rigid body about the fixed point. Then,

$$\vec{v_i} = \vec{\omega} \times \vec{r_i}$$

where $\vec{\omega}$ is the angular velocity of the rigid body. The equation 21 becomes

$$L = \sum_i m_i[\vec{r_i} \times (\vec{\omega} \times \vec{r_i})] \tag{22}$$

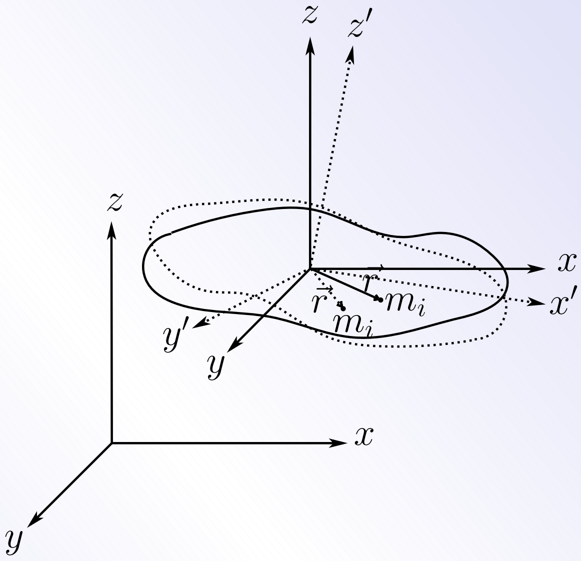

Figure 7: Rotation of a rigid body

[1] By using the vector triple product,

$$\begin{aligned}
\vec{L} &= \sum_i m_i\left[\vec{\omega}.(\vec{r_i}.\vec{r_i}) - \vec{r_i}.(\vec{r_i}.\vec{\omega})\right] \\
iL_x + jL_y + kL_z &= \sum_i m_i[(i\omega_x + j\omega_y + k\omega_z)(x_i^2 + y_i^2 + z_i^2) \\
&\quad -(ix_i + jy_i + kz_i)(x_i\omega_x + y_i\omega_y + z_i\omega_z)] \\
iL_x + jL_y + kL_z &= \sum_i m_i[(i\omega_x + j\omega_y + k\omega_z)(x_i^2 + y_i^2 + z_i^2) - (ix_i^2\omega_x + ix_iy_i\omega_y + ix_iz_i\omega_z \\
&\quad +jy_ix_i\omega_x + jy_i^2\omega_y + jy_iz_i\omega_z + kz_ix_i\omega_x + kz_iy_i\omega_y + kz_i^2\omega_z)] \qquad (23)
\end{aligned}$$

1

By using 23, the components of angular momentum can be written as

$$
\begin{aligned}
L_x &= \sum_i m_i[\omega_x(y_i^2 + z_i^2) - (x_i y_i \omega_y + x_i z_i \omega_z)] \\
L_y &= \sum_i m_i[\omega_y(x_i^2 + z_i^2) - (y_i x_i \omega_x + y_i z_i \omega_z)] \\
L_z &= \sum_i m_i[\omega_z(x_i^2 + y_i^2) - (z_i x_i \omega_x + z_i y_i \omega_y)]
\end{aligned}
$$

$$
\begin{aligned}
L_x &= \sum_i m_i(y_i^2 + z_i^2)\omega_x - \sum_i (m_i x_i y_i)\omega_y - \sum_i (m_i x_i z_i)\omega_z \\
L_y &= -\sum_i (m_i y_i x_i)\omega_x + \sum_i m_i(x_i^2 + z_i^2)\omega_y - \sum_i (m_i y_i z_i)\omega_z \\
L_z &= -\sum_i (m_i z_i x_i)\omega_x - \sum_i (m_i z_i y_i)\omega_y + \sum_i m_i(x_i^2 + y_i^2)\omega_z
\end{aligned}
\tag{24}
$$

Equation 24 is rewritten as

$$
\begin{aligned}
L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\
L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\
L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z
\end{aligned}
$$

$$
\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{25}
$$

$$
L = \mathbf{I}\omega \tag{26}
$$

where

$$
\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{bmatrix} \tag{27}
$$

The matrix I acts as a linear operator to transforms $\vec{\omega}$ into $\vec{L}$. It has elements that behave as the elements of a second-rank tensor. Therefore I is usually called the moment of inertia tensor. The elements of moment of inertia tensor can be written as

$$
I_{\alpha\beta} = \sum_i m_i(r_i^2 \delta_{\alpha_i\beta_i} - \alpha_i\beta_i) \qquad \begin{aligned} \delta_{\alpha_i\beta_i} &= 1 \quad for \quad \alpha_i = \beta_i \\ \delta_{\alpha_i\beta_i} &= 0 \quad for \quad \alpha_i \neq \beta_i \end{aligned} \tag{28}
$$

If the body is regarded as continuous with mass density $\rho(\vec{r})$ and $\sum_i m_i = \int \rho(\vec{r})dV$ then, the tensor can be written as

$$I_{\alpha\beta} = \int \rho(\vec{r})(r_i^2\delta_{\alpha_i\beta_i} - \alpha_i\beta_i)dV \qquad \begin{aligned} \delta_{\alpha_i\beta_i} &= 1 \;\; for \;\; \alpha_i = \beta_i \\ \delta_{\alpha_i\beta_i} &= 0 \;\; for \;\; \alpha_i \neq \beta_i \end{aligned} \tag{29}$$

[2] The kinetic energy of motion about a point is

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i v_i^2 \\ &= \frac{1}{2}\sum_i m_i \left[V_{CM} + (\vec{\omega}\times\vec{r_i})\right]^2 \\ &= \frac{1}{2}\sum_i m_i \left[V_{CM}^2 + 2V_{CM}(\vec{\omega}\times\vec{r_i}) + (\vec{\omega}\times\vec{r_i})^2\right] \\ T &= \frac{1}{2}\sum_i m_i V_{CM}^2 + V_{CM}\left(\vec{\omega}\times\sum_i m_i\vec{r_i}\right) + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r_i})^2 \end{aligned}$$

As distance $\vec{r_i}$ is measured from the CM, $\sum_i m_i\vec{r_i} = 0$ and $\sum_i m_i = M$. Then

$$\begin{aligned} T &= \frac{1}{2}MV_{CM}^2 + \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r_i})^2 \\ T &= T_t + T_r \end{aligned}$$

[2] $\vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B})$

where $T_t = MV_{CM}^2/2$ is the kinetic energy of the rigid body due to translational motion and T_r is the kinetic energy of the rigid body due to rotational motion. Thus,

$$\begin{aligned}
T_r &= \frac{1}{2}\sum_i m_i(\vec{\omega}\times\vec{r_i}).(\vec{\omega}\times\vec{r_i}) \\
&= \frac{1}{2}\sum_i m_i\vec{v_i}.(\vec{\omega}\times\vec{r_i}) \\
&= \frac{1}{2}\sum_i m_i\vec{\omega}.(\vec{r_i}\times\vec{v_i}) \\
T_r &= \frac{1}{2}\vec{\omega}.\sum_i(\vec{r_i}\times\vec{p_i}) \\
T_r &= \frac{1}{2}\vec{\omega}.\vec{L} = \frac{1}{2}\vec{\omega}.\mathbf{I}.\vec{\omega} = \frac{1}{2}\omega^2(\hat{n}.\mathbf{I}.\hat{n}) = \frac{1}{2}\omega^2 I \qquad (30)
\end{aligned}$$

where $\hat{n}$ is the unit vector in the ω direction and $I = [\tilde{n}]\mathbf{I}[n]$ is the moment of inertia of the body about the axis of rotation.

$$\begin{aligned}
I &= \frac{2T_r}{\omega^2} \\
&= \frac{1}{\omega^2}\sum_i m_i(\vec{r}\times\vec{\omega}).(\vec{r}\times\vec{\omega}) \\
I &= \sum_i m_i(\vec{r}\times\hat{n}).(\vec{r}\times\hat{n}) \qquad (31)
\end{aligned}$$

5. The eigenvalues of the inertia tensor and the principal axis transformation

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} \sum_i m_i(y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i y_i x_i & \sum_i m_i(x_i^2 + z_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i z_i x_i & -\sum_i m_i z_i y_i & \sum_i m_i(x_i^2 + y_i^2) \end{bmatrix}$$

From the equation above equation, we can see that,

$$I_{xy} = I_{yx} \qquad I_{yz} = I_{zy} \qquad I_{xz} = I_{zx}$$

i.e., the inertia tensor I generally have nine components, only six of them will be independent - the three along the diagonal plus three of the off-diagonal elements. According to equation 28,

$$I_{\alpha\beta} = \sum_i m_i(r_i^2 \delta_{\alpha_i\beta_i} - \alpha_i\beta_i) \qquad \delta_{\alpha_i\beta_i} = 1 \quad for \quad \alpha_i = \beta_i$$

$$\delta_{\alpha_i\beta_i} = 0 \quad for \quad \alpha_i \neq \beta_i$$

$$\begin{aligned} I_{xx} = I_x &= \sum_i m_i(r_i^2 - x_i^2) = \sum_i m_i(y_i^2 + z_i^2) \\ I_{yy} = I_y &= \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + z_i^2) \\ I_{zz} = I_z &= \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + y_i^2) \end{aligned}$$

The inertia coefficients depend both upon the location of the origin of the body set of axes and upon the orientation of these axes with respect to the body. This symmetry suggests that there exists a set of coordinates in which the tensor is diagonal with the three principal values I_1, I_2 and I_3.

If R is the rotation matrix $(\vec{r}\,' = R\vec{r})$, then the orthoganal transformation of the moment of inertial tensor I to I_D

$$\mathbf{I}_D \;=\; \tilde{R}\mathbf{I}R$$

This rotation can be expressed in terms of the Euler angles ϕ, θ, ψ as shown in equation 20. A proper choice of these angles will transform I into its diagonal form

$$\mathbf{I}_D \;=\; \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{32}$$

i.e., like any real symmetric matrix, MI tensor I can be diagonalized by choosing appropriate symmetry axes. These are called principal axes of I. The diagonal matrix I_D is called principal moment of inertia. The eigen values I_1, I_2 and I_3 are called components of principal moment of inertia. The directions of x', y' and z' defined by the rotation matrix R are called the *principal axes*, or *eigen vectors* of the inertia tensor.

With principal moments of inertia, the components of L would involve only the corresponding component of ω, thus

$$L_1 = I_1\omega_1 \qquad L_2 = I_2\omega_2 \qquad L_3 = I_3\omega_3$$

The kinetic energy of the rigid body with origin of body axes at its CM can be written as,

$$T = \frac{1}{2}MV_{CM}^2 + \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

We can understand the concept of principal axes through some geometrical considerations. The moment of inertia about a given axis has been defined in the equation 30 as

$$I = \hat{n}.\mathbf{I}.\hat{n}$$

where $\mathbf{I}$ is the inertia tensor and $\hat{n}$ is the unit vector along $\vec{\omega}$. Let the direction cosines of the axis be α, β and γ, then

$$\vec{n} = \alpha i + \beta j + \gamma k$$

$$\begin{aligned} I &= [\tilde{n}]\mathbf{I}[n] \\ &= \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \\ I &= I_{xx}\alpha^2 + I_{yy}\beta^2 + I_{zz}\gamma^2 + 2I_{xy}\alpha\beta + 2I_{yz}\beta\gamma + 2I_{zx}\alpha\gamma \qquad (33) \end{aligned}$$

It is convenient to define a vector $\vec{\rho}$ by the equation

$$\begin{aligned} \vec{\rho} &= \frac{\vec{n}}{\sqrt{I}} \\ i\rho_1 + j\rho_2 + k\rho_3 &= i\frac{\alpha}{\sqrt{I}} + j\frac{\beta}{\sqrt{I}} + k\frac{\gamma}{\sqrt{I}} \end{aligned} \tag{34}$$

The magnitude of $\vec{\rho}$ is thus related to the moment of inertia about the axis whose direction is given by $\vec{n}$.

On substituting the values of α, β and γ from equation 34 to 33,

$$1 = I_{xx}\rho_1^2 + I_{yy}\rho_2^2 + I_{zz}\rho_3^2 + 2I_{xy}\rho_1\rho_2 + 2I_{yz}\rho_2\rho_3 + 2I_{zx}\rho_1\rho_3 = \vec{\rho}.\mathbf{I}.\vec{\rho} \tag{35}$$

Equation 35 is the equation of an ellipsoid designated as the *inertial ellipsoid*.

We can transform ρ_1, ρ_2, ρ_3 to a set of Cartesian axes With the principal axes of the ellipsoid along the new Cartesian axes, the equation of an ellipsoid takes its normal form as

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1 \tag{36}$$

The principal moments of inertia I_1, I_2, I_3 determine the lengths of the axes of the inertia ellipsoid. If two of the roots of the secular equation are equal, the inertia ellipsoid thus has two equal axes and is an ellipsoid of revolution. If all three principal moments are equal, the inertia ellipsoid is a sphere.

If R_o is the radius of gyration,

$$I = MR_o^2$$

The vector $\vec{\rho}$ can be written as,

$$\vec{\rho} = \frac{\vec{n}}{R_o\sqrt{M}}$$

The radius vector to a point on the inertia ellipsoid is thus inversely proportional to the radius of gyration about the direction of the vector.

5.1. Classification of rigid bodies

1. If $I_1 \neq I_2 \neq I_3$, the rigid body is called asymmetric top.
2. If $I_1 = I_2 \neq I_3$, the body is called symmetric top.
3. If $I_1 = I_2 = I_3$, the body is called spherical top. Here one can choose any three mutually perpendicular axes as principal axes.

6. Theorems on moment of inertia

6.1. Theorem of perpendicular axes

The MI of a plane lamina about an axis perpendicular to its plane is equal to the sum of the moment of inertias about any two perpendicular axes in the plane that intersects the first axis.

The components of moments of inertia tensor can be written as

$$I_{\alpha\beta} = \sum_i m_i(r_i^2\delta_{\alpha_i\beta_i} - \alpha_i\beta_i) \qquad \delta_{\alpha_i\beta_i} = 1 \;\; for \;\; \alpha_i = \beta_i$$

$$\delta_{\alpha_i\beta_i} = 0 \;\; for \;\; \alpha_i \neq \beta_i$$

If the lamina is rotating about z axis with its CM at origin,

$$I_{zz} = I_z \;=\; \sum_i m_i(r_i^2 - y_i^2) = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i x_i^2 + \sum_i m_i y_i^2 = I_x + I_y$$

6.2. Theorem of parallel axis

The moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its CM, together with the product of its mass and the square of the distance between the axes. From the figure 8, $r_i = R + r_i'$. MI of the rigid body about

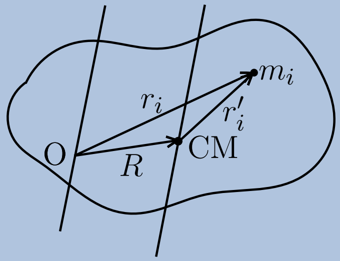

Figure 8: Rotation of a rigid body about an axis passing through O

an axis passing through O is

$$\begin{aligned} I &= \sum_i m_i (r_i \times \hat{n})^2 \\ &= \sum_i m_i [(R \times \hat{n}) + (r'_i \times \hat{n})]^2 \\ &= \sum_i m_i (R \times \hat{n})^2 + 2(R \times \hat{n}) \sum_i m_i (r'_i \times \hat{n}) + \sum_i m_i (r'_i \times \hat{n})^2 \end{aligned}$$

About the center of mass $\sum_i m_i r'_i = 0$,

$$\begin{aligned} I &= \sum_i m_i (R \times \hat{n})^2 + \sum_i m_i (r'_i \times \hat{n})^2 \\ I &= MR^2 sin^2\theta + I_{CM} \end{aligned}$$

where θ is the angle between $\vec{\omega}$ direction and $\vec{R}$ direction. That is the moment of inertia about a given axis is equal to the moment of inertia about a parallel axis through the center of mass plus the moment of inertia of the body, as if concentrated at the center of mass, with respect to the original axis.

7. The Euler equations of motion

For the rotational motion about a fixed point or the center of mass, the direct Newtonian approach leads to a set of equations known as Euler's equations of motion. For a rigid body, the general motion is both traslational and rotational. Therefore a frame attached to a rigid body is a non inertial frame. With reference to the fixed frame in space, the general operator equation can be written as

$$\left[\frac{d(\ )}{dt}\right]_{fixed} = \left[\frac{d(\ )}{dt}\right]_{body} + \vec{\omega} \times (\ )$$

where, ( ) contain any vector operator. Thus,

$$\left[\frac{d\vec{L}}{dt}\right]_{fixed} = \left[\frac{d\vec{L}}{dt}\right]_{body} + \vec{\omega} \times \vec{L}$$

$$N^{(e)} = \left[\frac{d\vec{L}}{dt}\right]_{body} + \vec{\omega} \times \vec{L} \tag{37}$$

The componets of equation 37 along x, y and z directions are

$$
\begin{aligned}
N_x &= \frac{d\vec{L}_x}{dt} + (\omega_y L_z - \omega_z L_y) \\
N_y &= \frac{d\vec{L}_y}{dt} + (\omega_z L_x - \omega_x L_z) \\
N_z &= \frac{d\vec{L}_z}{dt} + (\omega_x L_y - \omega_y L_x)
\end{aligned} \tag{38}
$$

If the body axes are taken as the principal axes, relative to reference point and if, I_1, I_2, I_3 are the principal moment of inertia along x, y, z directions, then

$$
L_x = I_1\omega_x \qquad L_y = I_2\omega_y \qquad L_z = I_3\omega_z
$$

The set of equations 38 becomes,

$$
\begin{aligned}
N_x &= I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z \\
N_y &= I_2\dot{\omega}_y - (I_3 - I_1)\omega_x\omega_z \\
N_z &= I_3\dot{\omega}_z - (I_1 - I_2)\omega_x\omega_y
\end{aligned} \tag{39}
$$

Equations 35 are Euler's equations of motion for a rigid body with one point fixed.

8. Torque-free motion of a rigid body

Euler's equations of motion for a rigid body with one point fixed are

$$\begin{aligned} N_x &= I_1\dot{\omega}_x - (I_2 - I_3)\omega_y\omega_z \\ N_y &= I_2\dot{\omega}_y - (I_3 - I_1)\omega_x\omega_z \\ N_z &= I_3\dot{\omega}_z - (I_1 - I_2)\omega_x\omega_y \end{aligned} \tag{40}$$

In the absence of any net torques, $(N = 0)$ they reduce to

$$\begin{aligned} I_1\dot{\omega}_x &= (I_2 - I_3)\omega_y\omega_z \\ I_2\dot{\omega}_y &= (I_3 - I_1)\omega_x\omega_z \\ I_3\dot{\omega}_z &= (I_1 - I_2)\omega_x\omega_y \end{aligned} \tag{41}$$

8.1. Poinsot's geometrical construction

Consider a coordinate system oriented along the principal axes of the body, but whose axes measure the components of a vector $\vec{\rho}$ along the instantaneous axis of rotation as

$$\vec{\rho} = \frac{\vec{n}}{\sqrt{I}} = \frac{\vec{\omega}}{\omega\sqrt{I}} = \frac{\vec{\omega}}{\sqrt{2T}} \tag{42}$$

In ρ space, we define a function $F(\rho)$ called surfaces of constant as

$$F(\rho) = \vec{\rho}.\mathbf{I}.\vec{\rho} \tag{43}$$

The surfaces of constatnt $F(\rho)$ are ellipsoid and $F = 1$ being the inertia ellipsoid. As the direction of the axis of rotation changes in time, the parallel vector $\vec{\rho}$ moves accordingly, its tip always defining a point on the inertia ellipsoid.

$$\nabla F(\rho) \;=\; \nabla(\vec{\rho}.\mathbf{I}.\vec{\rho}) \tag{44}$$

By using equation 35, equation 44 can be written as

$$\begin{aligned}
\nabla F(\rho) &= \left(i\frac{\delta}{\delta\rho_1} + j\frac{\delta}{\delta\rho_2} + k\frac{\delta}{\delta\rho_3}\right)(I_{xx}\rho_1^2 + I_{yy}\rho_2^2 + I_{zz}\rho_3^2 + 2I_{xy}\rho_1\rho_2 + 2I_{yz}\rho_2\rho_3 + 2I_{zx}\rho_1\rho_3) \\
&= 2i(I_{xx}\rho_1 + I_{xy}\rho_2 + I_{xz}\rho_3) + 2j(I_{yy}\rho_2 + I_{xy}\rho_1 + I_{yz}\rho_3) + 2k(I_{zz}\rho_3 + I_{yz}\rho_2 + I_{xz}\rho_1) \\
&= 2\begin{bmatrix} \rho_1 & \rho_2 & \rho_3 \end{bmatrix} \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix} \\
\nabla F(\rho) &= 2\vec{\rho}.\mathbf{I} = 2\mathbf{I}.\vec{\rho}
\end{aligned}$$

On substituting for $\vec{\rho}$ by using equation 42,

$$\nabla F(\rho) \;=\; = \frac{2\mathbf{I}.\vec{\omega}}{\sqrt{2T}} = \sqrt{\frac{2}{T}}\vec{L}$$

Thus, the $\vec{\omega}$ will always move such that the corresponding normal to the inertia ellipsoid is in the direction of the angular momentum $\vec{L}$. The tangent plane to the ellipsoid at the tip of $\vec{\rho}$ is called the *invariable plane* (Figure 9). The distance between the origin of the inertia ellipsoid and the inavrible plane is

$$\rho\, cos\,(\vec{\rho}.\vec{L}) \;=\; \frac{\vec{\rho}.\vec{L}}{L}$$

By using equation 42,

$$\frac{\vec{\rho}.\vec{L}}{L} \;=\; \frac{\vec{\omega}.\vec{L}}{L\sqrt{2T}} = \frac{\sqrt{2T}}{L}$$

Since both T and L are constants, the distance from the origin of the ellipsoid to the invariable plane remains constant and the point of contact is defined by the position of ρ. Thus the inertia ellipsoid rolls without slipping on the invariable plane. The curve traced out by the point of contact (tip of $\vec{\rho}$) on the inertia ellipsoid is known as the *polhode*, while the cnrve traced out by the point of contact (tip of $\vec{\rho}$) on the invariable plane is called the *herpolhode*.

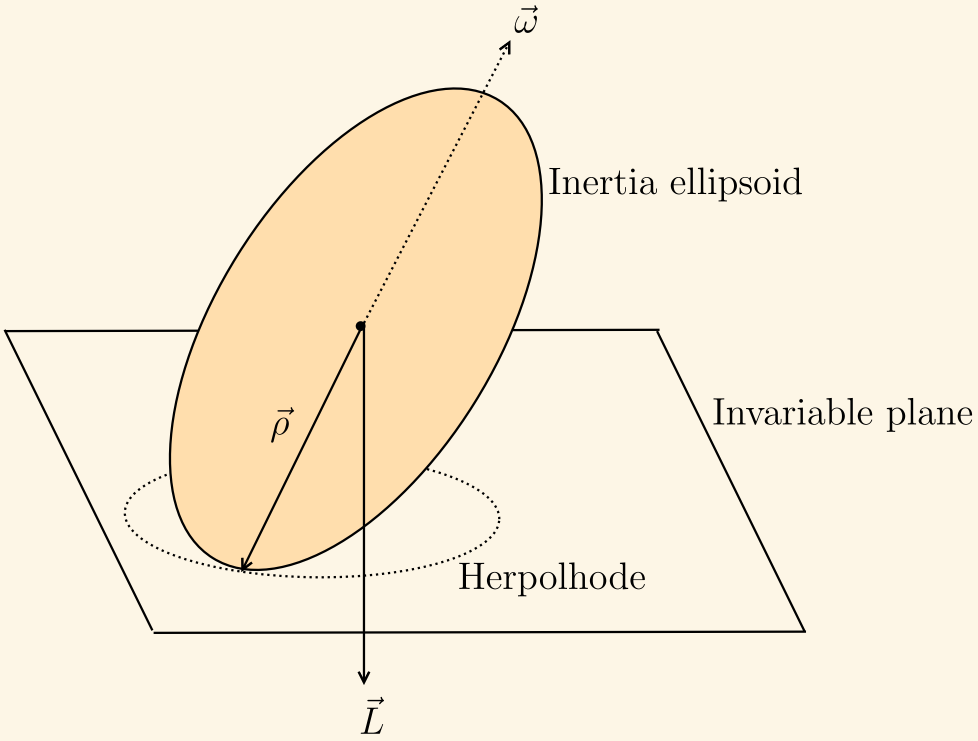

Figure 9: The motion of the inertia ellipsoid relative to the invariable plane.

8.2. Rotation of a symmetrical body

In case of a symmetrical body, the inertia ellipsoid is an ellipsoid of revolution, so that the polhode on the ellipsoid is clearly a circle about the symmetry axis. The herpolhode on the invariable plane is also a circle. An observer fixed in the body sees the angular velocity vector $\vec{\omega}$ move on the surface of a cone called the *body cone* whose intersection with the inertia ellipsoid is the polhode. An observer fixed in the space axes sees the $\vec{\omega}$ move on the surface of a space cone whose intersection with the invariable plane is the herpolhode. Thus, the free motion of the symmetrical rigid body is rolling of the body cone on the space cone.

If the moment of inertia about the symmetry axis is less than that about the other two principal axes, the inertia ellipsoid is prolate and the body cone rolls outside the space cone. When the moment of inertia about the symmetry axis is the greater, the ellipsoid is oblate and the body cone rolls inside the space cone.

In the absence of net torques, Euler's equations can be written as

$$\begin{aligned} I_1\dot{\omega}_x &= (I_2 - I_3)\omega_y\omega_z \\ I_2\dot{\omega}_y &= (I_3 - I_1)\omega_x\omega_z \\ I_3\dot{\omega}_z &= (I_1 - I_2)\omega_x\omega_y \end{aligned}$$

Let the symmetry axis of the body be taken as the L_z principal axis, so that $I_1 = I_2$.

Euler's equations reduce then to

$$I_1\dot{\omega}_x = (I_1 - I_3)\omega_y\omega_z \quad (45)$$

$$I_1\dot{\omega}_y = (I_3 - I_1)\omega_x\omega_z \quad (46)$$

$$I_3\dot{\omega}_z = 0 \implies \omega_z = constant$$

Equations 45 and 46 can be written as

$$\dot{\omega}_x = \frac{(I_1 - I_3)\omega_z}{I_1}\omega_y = \Omega\omega_y \quad (47)$$

$$\dot{\omega}_y = -\frac{(I_1 - I_3)\omega_z}{I_1}\omega_x = -\Omega\omega_x \quad (48)$$

where $\Omega = \frac{(I_1 - I_3)\omega_z}{I_1}$. On differentiating equation 47 and substituting for $\dot{\omega}_y$ by using equation 48,

$$\ddot{\omega}_x = \Omega\dot{\omega}_y = -\Omega^2\omega_x$$

$$\ddot{\omega}_x + \Omega^2\omega_x = 0 \quad (49)$$

Equation 49 is the standard diffeiential equation for simple harmonic motion. The solution of the equation can be written as

$$\omega_x = A\,cos\,\Omega t \quad (50)$$

$$\dot{\omega}_x = -\Omega\,A\,sin\,\Omega t \quad (51)$$

Equation 51 in equation 47,

$$\omega_x = A\,sin\,\Omega t \tag{52}$$

Equations 50 and 52 shows that the vector $i\omega_x + j\omega_y$ has a constant magnitude and rotates uniformly about the z axis of the body with the angular frequency Ω.

$$\texttt{\textbackslash omega} = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2} = \sqrt{A^2 + \omega_z^2} \tag{53}$$

Hence, the total angular velocity $\vec{\omega}$ is also constant in magnitude and precesses about the z axis with the same frequency, exactly as predicted by the Poinsot construction.

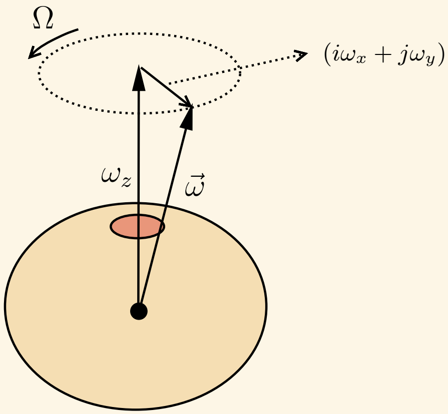

Figure 10: Precession of the angular velocity about the axis of symmetry in the force-free motion of a symmetrical rigid body.

9. Small oscillations of a mechanical system

The theory of small oscillations finds widespread physical applications in acoustics, molecular spectra, vibrations of mechanisms, and coupled electrical circuits. If the deviations of the system from stable equilibrium conditions are small enough, the motion can generally be described as that of a system of coupled linear harmonic oscillators.

An equilibrium position is classified as stable if a small disturbance of the system from equilibrium results only in small bounded motion about the rest position. The equilibrium is unstable if an infinitesimal disturbance eventually produces unbounded motion. If V is a minimum at equilibrium, any deviation from this position will produce an increase in V. By the conservation of energy, the velocities must then decrease and eventually come to zero, indicating bound motion. If V decreases as the result of some departure from equilibrium, the kinetic energy and the velocities increase indefinitely, corresponding to unstable motion.

9.1. Study of small oscillations using generalized coordinates

Consider scleronomic and holonomic conservative dynamic system having generalized coordinates $q_1, q_2, q_3, \ldots\ldots, q_n$. In the stable state kinetic energy of the system $T = 0$.

Then the Lagrangian L of the system is

$$L = T - V = -V \tag{54}$$

The Lagrangian equation is

$$-\frac{d}{dt}\left(\frac{\delta V}{\delta \dot{q}_i}\right) + \frac{\delta V}{\delta q_i} = 0, \qquad i = 1, 2, 3, ..., n. \tag{55}$$

If the configuration is initially at the equilibium position, with initial velocities $\dot{q}_i = 0$, then the system will continue in equilibrium indefinitely. The generalized coordinates at the equilibrium is written as $q_{10}, q_{20}, q_{30}, ..., q_{n0}$ and the equation 55 becomes

$$Q_i = -\left(\frac{\delta V}{\delta q_i}\right)_0 = 0 \tag{56}$$

Thus the potential energy has an extremum value at the equilibrium configuration of the system. For bound motion V is minimum at equilibrium. Let us disturb the system by giving small displacements η such that, $q_i = q_{oi} + \eta_i$, for the slightest possible perturbation, we can write

$$V(q_1, q_2, q_3, ..., q_n) > V(q_{01}, q_{02}, q_{03}, ..., q_{0n}) \tag{57}$$

By setting

$$V(q_{01}, q_{02}, q_{03}, ..., q_{0n}) = 0 \tag{58}$$

without losing any generality, the inequality 57 can be written as

$$V(q_1, q_2, q_3, ..., q_n) > 0$$

Then we expand a potential function using Taylor series around its minimum. We then obtain,

$$
\begin{aligned}
V(q_1, q_2, q_3, \ldots\ldots, q_n) &= V(q_{01}, q_{02}, q_{03}, \ldots\ldots, q_{0n}) + \sum_i \left(\frac{\delta V}{\delta q_i}\right)_0 \eta_i + \frac{1}{2}\sum_{ij}\left(\frac{\delta^2 V}{\delta q_i \delta q_j}\right)_0 \eta_i\eta_j \\
&\quad + \frac{1}{6}\sum_{ijk}\left(\frac{\delta^2 V}{\delta q_i \delta q_j \delta q_k}\right)_0 \eta_i\eta_j\eta_k + \ldots\ldots \qquad (59)
\end{aligned}
$$

By using equation 56 and 58 and by neglectiong the higher order terms, the equation 59 becomes

$$
\begin{aligned}
V(q_1, q_2, q_3, \ldots\ldots, q_n) &= \frac{1}{2}\sum_{ij}\left(\frac{\delta^2 V}{\delta q_i \delta q_j}\right)_0 \eta_i\eta_j \\
V &= \frac{1}{2}\sum_{ij} V_{ij}\eta_i\eta_j \qquad (60)
\end{aligned}
$$

where

$$
V_{ij} = \left(\frac{\delta^2 V}{\delta q_i \delta q_j}\right)_0
$$

The equation 60 can be written in the matrix form as

$$
\begin{aligned}
V &= \frac{1}{2}\begin{bmatrix} \eta_1 & \eta_2 & \ldots\ldots\ldots\ldots\eta_n \end{bmatrix}
\begin{bmatrix}
V_{11} & V_{12} & \ldots\ldots\ldots\ldots\ldots V_{1n} \\
V_{21} & V_{22} & \ldots\ldots\ldots\ldots\ldots V_{2n} \\
V_{31} & V_{32} & \ldots\ldots\ldots\ldots\ldots V_{3n} \\
. & & \\
. & & \\
. & & \\
V_{n1} & V_{n2} & \ldots\ldots\ldots\ldots\ldots V_{nn}
\end{bmatrix}
\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ . \\ . \\ . \\ \eta_n \end{bmatrix} \\
V &= \frac{1}{2}\tilde{\eta}\,\mathbf{V}\eta \qquad (61)
\end{aligned}
$$

where $\mathbf{V}$ is a tensor of $(n-1)$ rank. When the constraint equations are scleronomic, the kinetic energy of the system is given by

$$T = \frac{1}{2}\sum_i\sum_j m_{ij}\dot{q}_i\dot{q}_j \qquad \text{refer equations } \mathbf{??} \text{ and } \mathbf{??} \qquad (62)$$

where

$$m_{ij} = \sum_p m_p \frac{\delta\vec{r}_p}{\delta q_i}\frac{\delta\vec{r}_p}{\delta q_j} \qquad (63)$$

In the equation 63, p is summing over to number of particles. Since,

$$
\begin{aligned}
q_i &= q_{oi} + \eta_i \\
\dot{q}_i &= \dot{\eta}_i
\end{aligned}
$$

The equation 62 becomes,

$$T = \frac{1}{2}\sum_{ij} m_{ij}\dot{\eta}_i\dot{\eta}_j \tag{64}$$

From equation 63, we see that, $m_{ij} = m_{ij}(q_1, q_2, ..., q_n)$. We expand m_{ij} using Taylor series around the equilibrium position as,

$$m_{ij}(q_1, q_2, q_3, ..., q_n) = m_{ij}(q_{01}, q_{02}, q_{03}, ..., q_{0m}) + \sum_l \left(\frac{\delta m_{ij}}{\delta q_k}\right)_0 \eta_k + ... \tag{65}$$

As equation 64 is already quadratic in the $\dot{\eta}_i$, the lowest nonvanishing approximation to T is obtained by dropping all terms in equation 64 except first term. Therefore,

$$m_{ij}(q_1, q_2, q_3, ..., q_n) = m_{ij}(q_{01}, q_{02}, q_{03}, ..., q_{0n})$$

The equation 64 becomes,

$$T = \frac{1}{2}\sum_{ij} (m_{ij})_0\dot{\eta}_i\dot{\eta}_j$$

Denoting the constant values of the $(m_{ij})_0 = T_{ij}$, we can therefore write the kinetic energy as

$$T = \frac{1}{2}\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j \tag{66}$$

The equation 60 can be written in the matrix form as

$$T = \frac{1}{2}\left[\begin{array}{cccc} \dot{\eta}_1 & \dot{\eta}_2 & \dot{\eta}_3 & \ldots\ldots\ldots\ldots\dot{\eta}_n \end{array}\right] \left[\begin{array}{ccc} T_{11} & T_{12} & \ldots\ldots\ldots\ldots\ldots T_{1n} \\ T_{21} & T_{22} & \ldots\ldots\ldots\ldots\ldots T_{2n} \\ T_{31} & T_{32} & \ldots\ldots\ldots\ldots\ldots T_{3n} \\ . & & \\ . & & \\ . & & \\ T_{n1} & T_{n2} & \ldots\ldots\ldots\ldots\ldots T_{nn} \end{array}\right] \left[\begin{array}{c} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \\ . \\ . \\ . \\ \dot{\eta}_n \end{array}\right]$$

$$T = \frac{1}{2}\tilde{\dot{\eta}}\,\mathbf{T}\dot{\eta} \qquad (67)$$

where $\mathbf{T}$ is a tensor of $(n-1)$ rank. By using equations 60 and 66, the Lagrangian can be written as

$$L = \frac{1}{2}\left(\sum_{ij} T_{ij}\dot{\eta}_i\dot{\eta}_j - \sum_{ij} V_{ij}\eta_i\eta_j\right)$$

Taking η's as the general coordinates, the Lagrangian can be written as

$$L = \frac{1}{2}\left(\sum_{ij} T_{ij}\dot{\eta}_j^2 - \sum_{ij} V_{ij}\eta_j^2\right) \tag{68}$$
$$\frac{\delta L}{\delta \dot{\eta}_j} = \sum_{ij} T_{ij}\dot{\eta}_j$$
$$\frac{\delta L}{\delta \eta_j} = -\sum_{ij} V_{ij}\eta_j$$

By treating η_i as generalized coordinates and $\dot{\eta}_i$ as generalized velocities, the Lagrange's equations can be written as

$$\frac{d}{dt}\sum_{ij} T_{ij}\dot{\eta}_j + \sum_{ij} V_{ij}\eta_j = 0$$
$$\sum_{ij}(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j) = 0 \tag{69}$$

By using equations 61 and 67, the Lagrangian can also be written as

$$L = \frac{1}{2}\tilde{\dot{\eta}}\,\mathbf{T}\dot{\eta} - \frac{1}{2}\tilde{\eta}\,\mathbf{V}\eta \tag{70}$$

9.2. Normal coordinates and normal modes

Consider scleronomic and holonomic conservative dynamic system having generalized coordinates η_i. If the system is displaced slightly from equilibrium and then released, the system performs small oscillations about the equilibrium, the Lagrangian equations for the system are

$$\sum_{ij}(T_{ij}\ddot{\eta}_j + V_{ij}\eta_j) = 0 \tag{71}$$

The above equations of motion are linear differential equations with constant coefficients. The solution of the equations can be in the form

$$\eta_j = Ca_j e^{-i\omega t} \tag{72}$$

On substituting for $\ddot{\eta}_j$ and η_j by using equation 72 to the equation 71,

$$\begin{aligned} \sum_{ij}\left[T_{ij}(-\omega^2 Ca_j e^{-i\omega t}) + V_{ij}Ca_j e^{-i\omega t}\right] &= 0 \\ \sum_{ij}\left[V_{ij} - \omega^2 T_{ij}\right] a_j &= 0 \qquad (73) \end{aligned}$$

The equation 73 can be written in the matrix form as

$$\begin{bmatrix} V_{11}-\omega^2T_{11} & V_{12}-\omega^2T_{12} & \ldots\ldots\ldots\ldots\ldots V_{1n}-\omega^2T_{1n} \\ V_{21}-\omega^2T_{21} & V_{22}-\omega^2T_{22} & \ldots\ldots\ldots\ldots\ldots V_{2n}-\omega^2T_{2n} \\ V_{31}-\omega^2T_{31} & V_{32}-\omega^2T_{32} & \ldots\ldots\ldots\ldots\ldots V_{3n}-\omega^2T_{3n} \\ . & & \\ . & & \\ . & & \\ V_{m1}-\omega^2T_{n1} & V_{n2}-\omega^2T_{n2} & \ldots\ldots\ldots\ldots\ldots V_{nn}-\omega^2T_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ . \\ . \\ . \\ a_n \end{bmatrix} = 0$$

These n equations have a nontrivial solution if

$$\begin{bmatrix} V_{11}-\omega^2T_{11} & V_{12}-\omega^2T_{12} & \dots\dots\dots\dots V_{1m}-\omega^2T_{1n} \\ V_{21}-\omega^2T_{21} & V_{22}-\omega^2T_{22} & \dots\dots\dots\dots V_{2n}-\omega^2T_{2n} \\ V_{31}-\omega^2T_{31} & V_{32}-\omega^2T_{32} & \dots\dots\dots\dots V_{3n}-\omega^2T_{3n} \\ . & & \\ . & & \\ . & & \\ V_{n1}-\omega^2T_{n1} & V_{n2}-\omega^2T_{n2} & \dots\dots\dots\dots V_{nn}-\omega^2T_{nn} \end{bmatrix} = 0$$

$$\sum_{ij} V_{ij}-\omega^2T_{ij} = 0 \qquad (74)$$

The equation 74 gives polynomial in ω^2 of order n and has n roots. These roots are real and positive. These n roots may be all distinct or some of them may be same. If k root are identical, then we say that the system is k - fold degeneracy.

Thus the solution of the form given in the equation 72 not for one frequency, but in general for a set of n frequencies. A complete solution of the equations of motion therefore involves a superposition of oscillations with all the allowed frequencies. Thus, if the system is displaced slightly from equilibrium and then released, the system performs small oscillations about the equilibrium with the frequencies $\omega_1, \omega_2, \omega_3, \dots\dots\dots, \omega_n$. The solutions of the equation are therefore often designated as the *frequencies of free vibration* or as the *resonant frequencies* of the system.

The general solution of the equations of motion may be written as

$$\eta_i = C_k a_{jk} e^{-i\omega_k t} \tag{75}$$

where C_k is the complex scale factor for each frequency.

The solution for each generalized coordinate given in equation 75, is summing over all of the frequencies ω_k, satisfying the secular equation 74. Unless the frequencies are rational fractions of each other, η_i never repeats its initial value and is therefore not itself a periodic function of time. However, it is possible to transform from the η_i to a new set of generalized coordinates that are all simple periodic functions of time. These set of generalized coordinates are known as the *normal coordinates*. We define a new set of coordinates ξ_j as,

$$\eta_i = \sum_j a_{ij} \xi_j \tag{76}$$

Interms of matrix equation,

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ . \\ . \\ . \\ \eta_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots\dots\dots\dots a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots\dots a_{2n} \\ a_{31} & a_{32} & \dots\dots\dots\dots a_{3n} \\ . & & \\ . & & \\ . & & \\ a_{n1} & a_{n2} & \dots\dots\dots\dots a_{nn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ . \\ . \\ . \\ \xi_n \end{bmatrix}$$

$$\eta = \mathbf{A}\,\xi \quad \text{and} \quad \dot{\eta} = \mathbf{A}\,\dot{\xi} \tag{77}$$

where $\mathbf{A}$ is a tensor of $(n-1)$ rank. The potential energy is written in the matrix notation as

$$V = \frac{1}{2}\tilde{\eta}\mathbf{V}\eta \quad \text{refer equation 61}$$

By using equation 77,

$$V = \frac{1}{2}\widetilde{\mathbf{A}\xi}\mathbf{V}\mathbf{A}\xi$$

$$V = \frac{1}{2}\tilde{\xi}\tilde{\mathbf{A}}\mathbf{V}\mathbf{A}\xi \tag{78}$$

Since

$$\widetilde{\mathbf{A}}\mathbf{V}\mathbf{A} = \begin{bmatrix} V_{11} & 0 & 0 & \ldots\ldots\ldots\ldots\ldots 0 \\ 0 & V_{22} & 0 & \ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & V_{33} & \ldots\ldots\ldots\ldots\ldots 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & \ldots\ldots\ldots\ldots\ldots V_{nn} \end{bmatrix} = [\lambda] \tag{79}$$

Then the equation 78 reduces to

$$V = \frac{1}{2}\tilde{\xi}\lambda\xi \tag{80}$$

The kinetic energy is written in the matrix notation as

$$T = \frac{1}{2}\tilde{\dot{\eta}}\mathbf{T}\dot{\eta} \quad \text{refer equation 67}$$

By using equation 77,

$$T = \frac{1}{2}\widetilde{\mathbf{A}\dot{\xi}}\mathbf{T}\mathbf{A}\dot{\xi}$$

$$T = \frac{1}{2}\tilde{\dot{\xi}}\tilde{\mathbf{A}}\mathbf{T}\mathbf{A}\dot{\xi} \tag{81}$$

It was shown that

$$\tilde{\mathbf{A}}\mathbf{T}\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 1 & 0 & \ldots\ldots\ldots\ldots\ldots 0 \\ 0 & 0 & 1 & \ldots\ldots\ldots\ldots\ldots 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & \ldots\ldots\ldots\ldots\ldots 1 \end{bmatrix} = [\mathbf{1}] \qquad (82)$$

Then the equation 78 reduces to

$$T = \frac{1}{2}\tilde{\dot{\xi}}\dot{\xi}$$

$$= \frac{1}{2}\begin{bmatrix} \dot{\xi}_1 & \dot{\xi}_2 & \dot{\xi}_3 & \ldots\ldots\ldots\ldots\ldots\dot{\xi}_n \end{bmatrix} \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ . \\ . \\ . \\ \dot{\xi}_n \end{bmatrix}$$

$$T = \frac{1}{2}\left[\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2 + \ldots\ldots\ldots + \dot{\xi}_n^2\right] = \frac{1}{2}\sum \dot{\xi}_k^2 \qquad (83)$$

When the matrix $\mathbf{T} = [1]$, it can be shown that

$$[\lambda] \;=\; \begin{bmatrix} \omega_1^2 & 0 & 0 & \dots\dots\dots\dots\dots 0 \\ 0 & \omega_2^2 & 0 & \dots\dots\dots\dots\dots 0 \\ 0 & 0 & \omega_3^2 & \dots\dots\dots\dots\dots 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & \dots\dots\dots\dots\dots \omega_n^2 \end{bmatrix}$$

Then equation 80 can be written as

$$= \; \frac{1}{2}\left[\begin{array}{cccc} \xi_1 & \xi_2 & \xi_3 & \dots\dots\dots\dots\dots \xi_n \end{array}\right] \begin{bmatrix} \omega_1^2 & 0 & 0 & \dots\dots\dots\dots\dots 0 \\ 0 & \omega_2^2 & 0 & \dots\dots\dots\dots\dots 0 \\ 0 & 0 & \omega_3^2 & \dots\dots\dots\dots\dots 0 \\ . & & & \\ . & & & \\ . & & & \\ 0 & 0 & 0 & \dots\dots\dots\dots\dots \omega_n^2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ . \\ . \\ . \\ \xi_n \end{bmatrix}$$

$$V \;=\; \frac{1}{2}\left[\omega_1^2\xi_1^2 + \omega_2^2\xi_2^2 + \omega_3^2\xi_3^2 + \dots\dots\dots + \omega_n^2\xi_n^2\right] = \frac{1}{2}\sum \omega_k^2\xi_k^2 \tag{84}$$

By using equations 83 and 84, we can write the Lagrangian as

$$L = \frac{1}{2}(\dot{\xi}_k^2 - \omega_k^2\xi_k^2)$$
$$\frac{\delta L}{\delta\dot{\xi}_k} = \dot{\xi}_k \qquad \text{and} \qquad \frac{\delta \mathrm{L}}{\delta\xi_\mathrm{k}} = -\omega_\mathrm{k}^2\xi_\mathrm{k}$$

Then the Lagrangian equation can be written as

$$\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{\xi}_k}\right) - \frac{\delta L}{\delta\xi_k} = \ddot{\xi}_k + \omega_k^2\xi_k = 0 \tag{85}$$

The solution for the equation 85 can be written as

$$\xi_k = C_k e^{-i\omega_k t} \tag{86}$$

Thus from the equation 86, we can see that each normal cordinate ξ_k is simply periodic function involving only one of the resonant frequency. The modes of vibration corresponding to eaeh normal coordinates ξ_k are called *normal modes of vibration*. All of the particles in each mode vibrate with the same frequency and with the same phase, and the relative amplitudes being determined by the matrix elements a_{jk}. The complete motion is then built up out of the sum of the normal modes weighted with appropriate amplitude and phase factors contained in the $C_k's$

10. Free vibrations of CO_2 molecule

CO_2 molecule is a linear symmetrical triatomic molecule. In the equilibrium configuration of the molecule, two oxygen atoms are symmetrically located on each side of carbon atom as shown in the figure 11. All three atoms are on one straight line, the equilibrium distances apart being denoted by b. For simplicity, we shall first consider only vibrations along the line of the molecule ($y_i = 0, z_i = 0$), and the actual complicated interatomic potential will be approximated by two springs of force constant k joining the three atoms. There are three coordinates (x_1, x_2, x_3) marking the position of the three atoms on the line. In these coordinates, the potential energy is

$$V = \frac{k}{2}[(x_2 - x_1) - (x_{02} - x_{01})]^2 + \frac{k}{2}[(x_3 - x_2) - (x_{03} - x_{02})]^2$$
$$V = \frac{k}{2}[(x_2 - x_{02}) - (x_1 - x_{01})]^2 + \frac{k}{2}[(x_3 - x_{03}) - (x_2 - x_{02})]^2 \tag{87}$$

The η coordinates can be written as

$$\eta = x_i - x_{0i} \tag{88}$$

Equation 88 in equation 87,

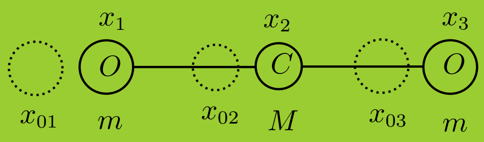

Figure 11: Model for a CO_2 molecule.

$$
\begin{aligned}
V &= \frac{k}{2}(\eta_2 - \eta_1)^2 + \frac{k}{2}(\eta_3 - \eta_2)^2 \\
&= \frac{k}{2}(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3) \\
&= \frac{1}{2}\begin{bmatrix} \eta_1 & \eta_2 & \eta_3 \end{bmatrix} \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \qquad (89)
\end{aligned}
$$

Hence the **V** tensor has the form

$$
\mathbf{V} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \qquad (90)
$$

The total kinetic energy of the molecule is

$$T = \frac{m}{2}\dot{\eta}_1^2 + \frac{M}{2}\dot{\eta}_2^2 + \frac{m}{2}\dot{\eta}_3^2 \tag{91}$$

where m and M are the masses of oxyzen and carbon molecules respectively. The equation 87 can be written in the matrix form as,

$$T = \frac{1}{2}\begin{bmatrix} \dot{\eta}_1 & \dot{\eta}_2 & \dot{\eta}_3 \end{bmatrix}\begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix}\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{bmatrix}$$

Then the **T** tensor becomes

$$\mathbf{T} = \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \tag{92}$$

By using equations 90 and 92, the secuar equation can be written as

$$[\mathbf{V} - \omega^2 \mathbf{T}] = \begin{bmatrix} k - \omega^2 m & -k & 0 \\ -k & 2k - \omega^2 M & -k \\ 0 & -k & k - \omega^2 m \end{bmatrix} = 0 \tag{93}$$

$$(k - \omega^2 m)\left[(2k - \omega^2 M)(k - \omega^2 m) - k^2\right] - k^2(k - \omega^2 m) = 0$$

$$(k - \omega^2 m)\left[(2k - \omega^2 M)(k - \omega^2 m) - 2k^2\right] = 0$$

$$\omega^2(k - \omega^2 m)\left[k(M + 2m) - \omega^2 mM\right] = 0 \tag{94}$$

The solutions of the equation 94 are

$$\begin{aligned} \omega^2 &= 0 &\implies&\quad \omega_1 = 0 \\ k - \omega^2 m &= 0 &\implies&\quad \omega_2 = \sqrt{\frac{k}{m}} \\ k(M + 2m) - \omega^2 mM &= 0 &\implies&\quad \omega_3 = \sqrt{\frac{k}{m}\left(1 + \frac{2m}{M}\right)} \end{aligned}$$

The first eigen value $\omega_1 = 0$ indicates that the molecule may be translated rigidly along its axis without any change in the potential energy. Since the restoring force against such motion is zero, the effective frequency must also vanish. We have made the assumption that the molecule has three degrees of freedom for vibrational motion, whereas in reality one of them is a rigid body degree of freedom.

The resonant frequency, ω_2 will be recognized as the well-known frequency of oscillation for a mass m suspended by a spring of force constant k. We are therefore led to expect that only the end atoms are vibrate and the center atom remains stationary in this mode. It is only in the third mode of vibration, ω_3,

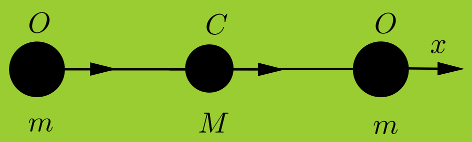

Figure 12: Mode1: Translational motion of CO_2 molecule.

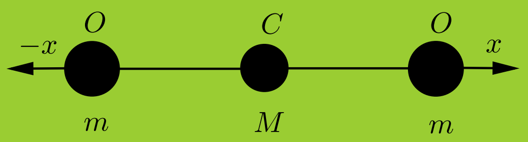

Figure 13: Mode2: Symmetric stretching of CO_2 molecule.

that the mass M can participate in the oscillatory motion. These predictions are verified by examining the eigenvectors for the three normal modes. The

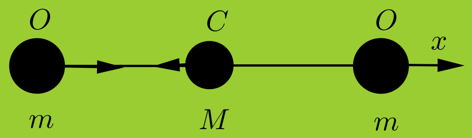

Figure 14: Mode3: Asymmetric stretching of CO_2 molecule.

eigen vectors are given by

$$\begin{aligned}
\left[\mathbf{V} - \omega_j^2 \mathbf{T}\right] a_{ij} &= 0 \\
\begin{bmatrix} k - \omega_j^2 m & -k & 0 \\ -k & 2k - \omega_j^2 M & -k \\ 0 & -k & k - \omega_j^2 m \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} &= 0 \\
(k - \omega_j^2 m) a_{1j} - k a_{2j} &= 0 \qquad (95) \\
-k a_{1j} + (2k - \omega_j^2 M) a_{2j} - k a_{3j} &= 0 \qquad (96) \\
-k a_{2j} + (k - \omega_j^2 m) a_{3j} &= 0 \qquad (97)
\end{aligned}$$

The normalization condition is given by

$$\begin{aligned} \tilde{\mathbf{A}}\mathbf{T}\mathbf{A} &= 1 \\ \begin{bmatrix} a_{1j} & a_{2j} & a_{3j} \end{bmatrix} \begin{bmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{bmatrix} &= 1 \\ ma_{1j}^2 + Ma_{2j}^2 + ma_{3j}^2 &= 1 \end{aligned} \tag{98}$$

For $\omega_1 = 0$, from equations 95 and 97,

$$a_{11} = a_{21} = a_{31} \tag{99}$$

By using equation 99 in 98,

$$a_{11} = a_{21} = a_{31} = \frac{1}{\sqrt{2m + M}} \tag{100}$$

In the second mode $k - \omega_2^2 m = 0$, from equation 95, $a_{22} = 0$, then from equation 96, $a_{12} = -a_{32}$. The equation 98 gives,

$$a_{12} = \frac{1}{\sqrt{2m}} \quad \text{and} \quad \mathrm{a}_{32} = -\frac{1}{\sqrt{2\mathrm{m}}} \tag{101}$$

For ω_3, from equation 95 and 97, $a_{13} = a_{33}$. In the third mode $\omega_3^2 = \frac{k}{m}\left(1 + \frac{2m}{M}\right)$, the equation 96 becomes,

$$
\begin{aligned}
-ka_{13} + \left[2k - \frac{k}{m}\left(1 + \frac{2m}{M}\right)\right] a_{23} - ka_{13} &= 0 \\
a_{23} &= -\frac{2m}{M} a_{13} \qquad (102)
\end{aligned}
$$

Using $a_{13} = a_{33}$, and equation 102 in equation 98,

$$
\begin{aligned}
2ma_{13}^2 + M\left(-\frac{2m}{M} a_{13}\right)^2 &= 1 \\
a_{13} &= \frac{1}{\sqrt{2m\left(1 + \frac{2m}{M}\right)}} \qquad (103)
\end{aligned}
$$

Equation 103 in equation 102,

$$
a_{23} = -\frac{2m}{M} \frac{1}{\sqrt{2m\left(1 + \frac{2m}{M}\right)}} = \frac{-2}{\sqrt{2M\left(2 + \frac{M}{m}\right)}} \qquad (104)
$$

Then the $\mathbf{A}$ matrix can be written as,

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2m+M}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \\ \frac{1}{\sqrt{2m+M}} & 0 & \frac{-2}{\sqrt{2M\left(2+\frac{M}{m}\right)}} \\ \frac{1}{\sqrt{2m+M}} & -\frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \end{bmatrix}$$

The $\mathbf{A}^\dagger$ can be evaluated as

$$\mathbf{A}^\dagger = \begin{bmatrix} \frac{m}{\sqrt{2m+M}} & \frac{M}{\sqrt{2m+M}} & \frac{m}{\sqrt{2m+M}} \\ \sqrt{\frac{m}{2}} & 0 & -\sqrt{\frac{m}{2}} \\ \sqrt{\frac{mM}{2(2m+M)}} & -\sqrt{\frac{2mM}{(2m+M)}} & \sqrt{\frac{mM}{2(2m+M)}} \end{bmatrix}$$

The normal coordiantes are given by

$$\xi = \mathbf{A}^{\dagger}\eta \quad (105)$$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \frac{m}{\sqrt{2m+M}} & \frac{M}{\sqrt{2m+M}} & \frac{m}{\sqrt{2m+M}} \\ \sqrt{\frac{m}{2}} & 0 & -\sqrt{\frac{m}{2}} \\ \sqrt{\frac{mM}{2(2m+M)}} & -\sqrt{\frac{2mM}{(2m+M)}} & \sqrt{\frac{mM}{2(2m+M)}} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \quad (106)$$

$$\xi_1 = \frac{1}{\sqrt{2m+M}}\left[m(\eta_1+\eta_3)+M\eta_2\right] \quad (107)$$

$$\xi_2 = \sqrt{\frac{m}{2}}(\eta_1-\eta_3) \quad (108)$$

$$\xi_3 = \sqrt{\frac{mM}{2m+M}}\left[(\eta_1+\eta_3)-\sqrt{2}\eta_2\right] \quad (109)$$

Any general longitudinal vibration of the molecule will be linear combination of the normal modes ω_2 and ω_3. The amplitudes of the normal modes, and their phases relative to each other can be determined by the initial conditions

11. Normal modes of double pendulum

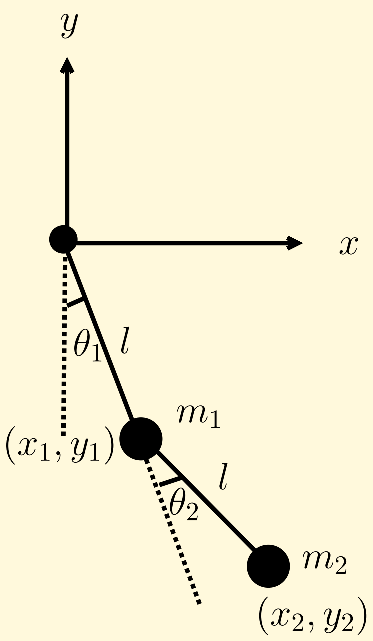

Figure 15: Double pendulum.

From the figure 15 we can write

$$
\begin{aligned}
x_1 &= l\sin\,\theta_1 & y_1 &= -l\cos\,\theta_2\\
\dot{x_1} &= l\cos\,\theta_1\dot{\theta}_1 & \dot{y_1} &= l\sin\,\theta_1\dot{\theta}_1\\
x_2 &= x_1 + l\sin\,(\theta_1+\theta_2) & y_2 &= y_1 - l\cos\,(\theta_1+\theta_2)\\
\dot{x_2} &= \dot{x_1} + l\cos\,(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) & \dot{y_2} &= \dot{y_1} + l\sin\,(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)\\
\dot{x_2} &= l\cos\,\theta_1\dot{\theta}_1 + l\cos\,(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) & \dot{y_2} &= l\sin\,\theta_1\dot{\theta}_1 + l\sin\,(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2
\end{aligned}
$$

The kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}m(\dot{x_1}^2 + \dot{y_i}^2 + \dot{x_2}^2 + \dot{y_2}^2)\\
&= \frac{1}{2}ml^2[2\dot{\theta_1}^2 + 2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2)\cos\,\theta_2 + (\dot{\theta}_1+\dot{\theta}_2)^2]
\end{aligned}
$$

The potential energy is

$$
\begin{aligned}
V &= mg(y_1+y_2)\\
&= -mgl[2\cos\,\theta_1 + \cos\,(\theta_1+\theta_2)]\\
&= mgl\left[4\sin^2\,\frac{\theta_1}{2} + 2\sin^2\,\frac{(\theta_1+\theta_2)}{2}\right]
\end{aligned}
$$

For small oscillations sin $\theta \to \theta$ and cos $\theta \to 1$, then T and V expressions can be written as

$$\begin{aligned}
T &= \frac{1}{2}ml^2[2\dot{\theta}_1{}^2 + 2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2) + (\dot{\theta}_1 + \dot{\theta}_2)^2] \\
&= ml^2\left[\frac{5}{2}\dot{\theta}_1{}^2 + 2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2{}^2\right] \\
&= ml^2\left[\begin{array}{cc} \dot{\theta}_1 & \dot{\theta}_2 \end{array}\right]\left[\begin{array}{cc} \frac{5}{2} & 1 \\ 1 & \frac{1}{2} \end{array}\right]\left[\begin{array}{c} \dot{\theta}_1 \\ \dot{\theta}_2 \end{array}\right] \\
V &= mgl\left[\theta_1^2 + \frac{(\theta_1 + \theta_2)^2}{2}\right] \\
&= \frac{mgl}{2}\left[3\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2\right] \\
&= \frac{mgl}{2}\left[\begin{array}{cc} \theta_1 & \theta_2 \end{array}\right]\left[\begin{array}{cc} 3 & 1 \\ 1 & 1 \end{array}\right]\left[\begin{array}{c} \theta_1 \\ \theta_2 \end{array}\right]
\end{aligned}$$

Then

$$[\mathbf{V} - \omega^2\mathbf{T}] = \frac{mgl}{2}\left[\begin{array}{cc} 3 & 1 \\ 1 & 1 \end{array}\right] - \omega^2\,\frac{ml^2}{2}\left[\begin{array}{cc} 5 & 2 \\ 2 & 1 \end{array}\right] = 0$$

On substituting $\omega^2 = \lambda g/l$

$$
\begin{aligned}
[\mathbf{V} - \omega^2 \mathbf{T}] &= \frac{mgl}{2} \begin{bmatrix} 3-5\lambda & 1-2\lambda \\ 1-2\lambda & 1-\lambda \end{bmatrix} = 0 \\
&= \frac{mgl}{2}(\lambda^2 - 4\lambda + 2) = 0 \\
\lambda &= 2 \pm \sqrt{2} \\
\lambda_1 &= 2 - \sqrt{2}, \qquad \lambda_2 = 2 + \sqrt{2} \\
\omega_1^2 &= (2 - \sqrt{2})\frac{g}{l}, \qquad \omega_2^2 = (2 + \sqrt{2})\frac{g}{l}
\end{aligned}
$$