Classical Mechanics

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1. Constraints

Constraints means restrictions; constrained motion means restricted motion. Most of the motion that we encounter, is constrained motion. Most physical realizations of constrained motion involve surfaces of other bodies, for example,

1. **Motion of a billiard ball on the table:** Motion of a billiard ball is restricted by the boundaries of the table, and it moves on the surface of the table. If the centre of mass of a billiard ball of radius R moving on a billiard table of length $2a$ and breadth $2b$, must satisfy the relation

$$-a + R \leq x \leq a - R, \qquad -b + R \leq y \leq b - R, \qquad z = R$$

assuming that the origin of the coordinate axes is at the centre of the rectangular table and x and y axes are parallel to length and breadth respectively. i.e., a set of one equation and two inequalities, defines the motion of a billiard ball at all instants of time.

2. **The motion of a simple pendulum:** The bob of the pendulum moves in a vertical plane (say zx plane). Its distance from the fulcrum is fixed. Thus, if (x, y, z) is coordinate of the bob then,

$$y = constant, \qquad z^2 + x^2 = l^2$$

are the restrictions on the coordinates of the bob.

Physically, constrained motion is realised by the forces which arise when the object in motion is in contact with the constraining surfaces or curves. These forces, called constraint forces, are usually stiff elastic forces at the contact. If there are no constraints, motion of the particle is described by the trajectory $\vec{r}(t) = ix + jy + kz$ and by its momentum $\vec{p}(t) = ip_x + jp_y + kp_z$. Thus the position of the particle is specified by three coordinates. If there are N particles, $3N$ independent coordinates are necessary for the position specification of the system at a time t. Presence of constraints may reduce the number of independent variables.

1.1. Classification of Constraints

1. (a) **Scleronomic:** constraint relations do not explicitly depend on time,

   (b) **Rheonomic:** constraint relations depend explicitly on time,

2. (a) **Holonomic:** conditions of constraint can be expressed as equations connecting the coordinates of the particles,

   (b) **Non holonomic:** constraint relations are not holonomic,

3. (a) **Conservative:** total mechanical energy of the system is conserved while performing, the constrained motion. Constraint forces do not do any work,

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(b) **Dissipative:** constraint forces do work and total mechanical energy is not conserved.

4. (a) **Bilateral:** at any point on the constraint surface both the forward and backward motions are possible. Constraint relations are not in the form of inequalities but are in the form of equations,

   (b) **Unilateral:** at some points no forward motion is possible. Constraint relations are expressed in the form of inequalities.

1.2. Holonomic and non holonomic constraints

If one can write the equations of constraints as

$$
\begin{aligned}
f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
&. \\
&. \\
f_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
f_{i+1}(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
&. \\
&. \\
f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0
\end{aligned} \tag{1}
$$

where $k < n$, then such constraints are known as holonomic constraints. The constraints which cannot be expressed in the form of algebraic equations are non holonomic constraints, however, they could be expressed as inequalities.

1.3. Examples of constraints

1. **Rigid body:** A rigid body is a system of particles such that the distance between any pair of particles remains constant in time. Thus the motion of a rigid body is constrained by the equations

$$\vec{r}_i - \vec{r}_k = const. \tag{2}$$

where the pair of subscripts $(i\ ,\ k)$ run over all distinct pairs of particles forming the body. Obviously this constraint is scleronomic. The constraint is also holonomic and bilateral. The constraint relations 2 can be written as

$$|\vec{r}_i - \vec{r}_k|^2 = const.$$

Taking differentials

$$(\vec{r}_i - \vec{r}_k).\Delta(\vec{r}_i - \vec{r}_k) = 0 \tag{3}$$

Work done by the system is

$$W = \sum_i \sum_k (\vec{F}_{ik}.\Delta\vec{r}_i + \vec{F}_{ki}.\Delta\vec{r}_k) \tag{4}$$

Let the internal force of constraint on the i^{th} particle due to the k^{th} particle be represented by $\vec{F}_{ik}$. By Newton's third law we have,

$$\vec{F}_{ik} = -\vec{F}_{ki} \tag{5}$$

Thus we have for the work done by $\vec{F}_{ik}$ due to a displacement $\Delta\vec{r}_i$ of the i^{th} particle,

$$\vec{F}_{ik}.\Delta\vec{r}_i = -\vec{F}_{ki}.\Delta\vec{r}_k \tag{6}$$

On combining equations 4 and 6 we can write the total work done by the system

$$W = \sum_i \sum_k \vec{F}_{ik}.(\Delta\vec{r}_i - \Delta\vec{r}_k) \tag{7}$$

Since all $\vec{F}_{ik}$ are the internal forces which arise purely due to interaction between all possible pairs of particles, it is only natural that $\vec{F}_{ik}$ will act parallel to the line joining the i^{th} and k^{th} particles. Thus we can write,

$$\vec{F}_{ik} = C_{ik}(\vec{r}_i - \vec{r}_k) \tag{8}$$

where C_{ik}'s are real constants and symmetric in i and k. Substituting in the above expression for the total work, we have

$$W = \sum_i \sum_k C_{ik}(\vec{r}_i - \vec{r}_k).(\Delta\vec{r}_i - \Delta\vec{r}_k) \tag{9}$$

In equation 9 each individual term of the summand is zero. Thus the constraint of rigidity is conservative in nature, apart from its being scleronomic, holonomic and bilateral.

2. **Deformable bodies:** Suppose that the deformation of the body is changing in time according to a certain prescribed function of time. Then the motion of such a body is constrained by the equation

$$|\vec{r}_i - \vec{r}_k| = f(t) \tag{10}$$

where $\vec{r}_i$ and $\vec{r}_k$ are position vectors and the pair of subscripts $(i\ ,\ k)$ runs over all distinct pairs of particles in the body. These constraint relations cannot give the total work $W = 0$. Hence this is a rheonomic, holonomic, bilateral and dissipative constraint.

3. **Gas in a spherical container of radius R**. Here if $\vec{r}_i$ is a position vector of the i^{th} gas molecule (origin is at the centre of the sphere) then

$$x_i^2 + y_i^2 + z_i^2 \leq R^2 \tag{11}$$

Thus, we have a constraint equation given by an inequality and hence is non holonomic constraint.

4. **Rolling without sliding:** Suppose a spherical ball is rolling on a plane without sliding. We assume that the surfaces in contact are perfectly rough. Thus the frictional forces are not negligible. Since the point of contact is not sliding, the frictional forces do not do any work, and hence the total mechanical energy of

the rolling body is conserved. Thus the constraint is conservative. To obtain the constraint equation we note that rolling without sliding means that the relative velocity of the point of contact with respect to the plane is zero. Then the velocity v of any point P in the rolling body, as seen from a fixed frame of reference, is given by

$$v = V_{CM} + \vec{\omega} \times \vec{r} \tag{12}$$

where V_{CM} is the velocity of the centre of mass and $\vec{r}$ is measured from the CM to the point P under consideration. Thus the velocity of the point of contact is obtained by putting $\vec{r} = -r\hat{n}$ in equation 12, where $\hat{n}$ is the unit vector along the outward normal to the plane and r is the radius of the sphere. Since there is no sliding of this point we must have the instantaneous velocity v at the contact

$$v = V_{CM} - r(\vec{\omega} \times \hat{n}) = 0 \tag{13}$$

For a sphere this constraint is non integrable because ω is generally not expressible in the form of a total time derivative of any single coordinate. Thus the constraint is non holonomic. However, for a cylinder, $\omega = d\theta/dt$ where θ is the angle of rotation of the cylinder about its axis. Therefore this equation of constraint can be integrated and reduced to a holonomic form, giving a relation between r and the coordinates of the centre of mass.

2. Generalized coordinates

The problem of system of n particles can be solved when the number of constraint equations are less than *3n*. Let there be k equations of constraints $k < 3n$,

$$\begin{aligned} f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\ f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\ &. \\ &. \\ &. \\ &. \\ f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \qquad (14) \end{aligned}$$

i.e., $3n - k$ coordinates may be regarded as free and which define the position of the system at any moment of time t. Then the number of independent coordinate to specify the motion at a given time t is $3n - k$. These independent coordinates are called *degrees of freedom*.

In the case of a free material particle, for instance, $n = 1$ and $k = 0$ so that it has $3n - k = 3$ degrees of freedom. If the particle is constrained to move on a surface whose equation may be taken as $f(x, y, z) = 0 (z = 0)$, we clearly have $k = 1$ and therefore it would have $3n - k = 2$ degrees of freedom. On the other hand, for a dumb-bell shaped structure, with two particles connected by a rod of length l; a constraint equation becomes

$$f(x, y, z) = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0$$

we have $n = 2$ and $k = 1$, therefore it has $3n - k = 5$ degrees of freedom. The degrees of freedom are represented by $3n - k$ variables, $q_1, q_2, , q_{n-k}$. The old coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n$ are expressed in terms of $q's$ as,

$$\begin{aligned} \vec{r}_1 &= \vec{r}_1(q_1, q_2, , q_{n-k}; t) \\ \vec{r}_2 &= \vec{r}_2(q_1, q_2, , q_{n-k}; t) \\ &. \\ &. \\ &. \\ &. \\ \vec{r}_n &= \vec{r}_n(q_1, q_2, , q_{n-k}; t) \end{aligned} \tag{15}$$

2.1. Virtual displacement

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\vec{r}_i$, consistent with the forces and constraints imposed on the system at the given instant of time t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing.

2.2. Virtual work

Total work done by the external forces when virtual displacements are made in n particle system, is known as virtual work. If $\vec{F}_i^{(a)}$ be the applied force and $\vec{f}_i$ be the constraint force acting on i_{th} particle, the net force acting on the system is

$$\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i \tag{16}$$

If $\delta\vec{r}_i$ is the virtual displacement, the work done on the system is

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = \sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i \tag{17}$$

When system is in equilibrium

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = 0 \tag{18}$$

Thus equation 17 reduces to

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i = 0 \tag{19}$$

The virtual displacements $\delta\vec{r}_i$ are such that the constraint forces do no work ($\sum_i \vec{f}_i.\delta\vec{r}_i = 0$). Thus,

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i = 0 \tag{20}$$

i.e., The condition for static equilibrium is that the virtual work done by all the applied forces should vanish, provided the virtual work done by all the constraint forces vanishes. This is called the principle of virtual work.

2.3. D'Alembert's Principle

Consider the motion of n particle system. Then, by Newton's law,

$$\sum_i \vec{F}_i = \sum_i \dot{\vec{p}}_i \tag{21}$$

Combining equations 21 and 16,

$$\begin{aligned} \sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i &= \sum_i \dot{\vec{p}}_i \\ \sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i - \sum_i \dot{\vec{p}}_i &= 0 \end{aligned} \tag{22}$$

The equation 22 states that the particles in the system will be in equilibrium under a force equal to the actual force plus a *reversed effective force* $-\dot{\vec{p}}_i$. The work done now can be written as

$$\sum_i (\vec{F}_i^{(a)} + \vec{f}_i - \dot{\vec{p}}_i).\delta\vec{r}_i \;=\; 0 \tag{23}$$

The virtual displacements $\delta\vec{r}_i$ are such that the constraint forces do no work ($\sum_i \vec{f}_i.\delta\vec{r}_i = 0$). Thus,

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i).\delta\vec{r}_i \;=\; 0 \tag{24}$$

The equation 24 is called D'Alembert's Principle. D'Alembert's principle does not involve forces of constraint. i.e., any dynamical problem could be converted into an effective static problem.

3. Lagrange's Equations of the second kind

In D'Alembert's principle, the virtual displacements $\delta\vec{r}_i$ are not independent. Therefore, the D'Alembert's equation is a single equation. If the constraints are holonomic, we use independent set of variables $\{q_i\}$. When this is done, we get, n equations, one each for each q. These equations are Lagrange's equations. For n particle system the D'Alembert's equation is,

$$\sum_i \left(\vec{F}_i^{(a)} - \dot{\vec{p}}_i\right).\delta\vec{r}_i = 0 \tag{25}$$

If we have n particles at $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n$ and there are k equations of holonomic constraints, then there are $3N - k = m$ generalized coordinates. They are denoted by $q_1, q_2, ..., q_m$. Thus we have,

$$\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_m; t) \tag{26}$$

Virtual displacement $\delta\vec{r}_i$ are not independent and does not involve time. Thus,

$$\delta\vec{r}_i = \frac{\delta\vec{r}_i}{\delta q_1}\delta q_1 + \frac{\delta\vec{r}_i}{\delta q_2}\delta q_2 + \frac{\delta\vec{r}_i}{\delta q_3}\delta q_3 + ... + \frac{\delta\vec{r}_i}{\delta q_m}\delta q_m$$

$$\delta\vec{r}_i = \sum_{j=1}^{m} \frac{\delta\vec{r}_i}{\delta q_j}\delta q_j \tag{27}$$

Here, δq_j are virtual displacement of generalized coordinates. Velocity of the i^{th} particle, v_i is given as (on differentiating the equation 26),

$$\frac{d\vec{r}_i}{dt} = \frac{\delta\vec{r}_i}{\delta q_1}\frac{\delta q_1}{\delta t} + \frac{\delta\vec{r}_i}{\delta q_2}\frac{\delta q_2}{\delta t} + \frac{\delta\vec{r}_i}{\delta q_3}\frac{\delta q_3}{\delta t} + \dots\dots + \frac{\delta\vec{r}_i}{\delta q_m}\frac{\delta q_m}{\delta t} + \frac{\delta\vec{r}_i}{\delta t}$$

$$v_i = \sum_{j=1}^{m}\frac{\delta\vec{r}_i}{\delta q_j}\frac{\delta q_j}{\delta t} + \frac{\delta\vec{r}_i}{\delta t}$$

$$v_i = \sum_{j=1}^{m}\frac{\delta\vec{r}_i}{\delta q_j}\dot{q}_j + \frac{\delta\vec{r}_i}{\delta t} \tag{28}$$

$$\frac{\delta v_i}{\delta\dot{q}_j} = \frac{\delta\vec{r}_i}{\delta q_j} \tag{29}$$

The virtual work of the system is

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = \sum_i\sum_j \vec{F}_i.\frac{\delta\vec{r}_i}{\delta q_j}\delta q_j \qquad \text{(by using equation 27)}$$

$$\sum_i \vec{F}_i.\delta\vec{r}_i = \sum_j Q_j\delta q_j \tag{30}$$

where the Q_j are called the components of the generalized force, defined as

$$Q_j = \sum_i \vec{F}_i.\frac{\delta\vec{r}_i}{\delta q_j} \tag{31}$$

Note that $q's$ need not have the dimensions of length, so the $Q's$ do not necessarily have the dimensions of force, but $Q_j\delta q_j$ must always have the dimensions of work. For example, Q_j might be a torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j d\theta_j$ a differential of work.

Again consider the equation 30,

$$\begin{aligned}
\sum_j Q_j \delta q_j &= \sum_i \vec{F}_i.\delta\vec{r}_i \\
&= \sum_i \dot{\vec{p}}_i.\delta\vec{r}_i \\
&= \sum_i m_i \ddot{\vec{r}}_i.\delta\vec{r}_i \\
&= \sum_i \sum_j m_i \ddot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\delta q_j \\
\sum_j Q_j \delta q_j &= \sum_i \sum_j \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\right) - m_i \dot{\vec{r}}_i.\frac{d}{dt}\left(\frac{\delta\vec{r}_i}{\delta q_j}\right)\right]\delta q_j \qquad (32)
\end{aligned}$$

We can see from the equation 32, $\vec{r}_i$ is differentiable with respect to both t and q_j, we can interchange the differentiation with respect to t and q_j in equation 32.

$$\begin{aligned}
\sum_j Q_j \delta q_j &= \sum_i \sum_j \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\right) - m_i \dot{\vec{r}}_i.\frac{\delta}{\delta q_j}\left(\frac{d\vec{r}_i}{dt}\right)\right]\delta q_j \\
&= \sum_i \sum_j \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\right) - m_i \dot{\vec{r}}_i.\frac{\delta}{\delta q_j}\left(\frac{d\vec{r}_i}{dt}\right)\right]\delta q_j \\
&= \sum_i \sum_j \left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i.\frac{\delta\vec{r}_i}{\delta q_j}\right) - m_i \dot{\vec{r}}_i.\frac{\delta\vec{v}_i}{\delta q_j}\right]\delta q_j \qquad (33)
\end{aligned}$$

From the equation 29, we can write

$$\frac{\delta v_i}{\delta \dot{q}_j} = \frac{\delta \vec{r}_i}{\delta q_j} \tag{34}$$

Equation 34 in equation 33 gives,

$$\begin{aligned}
\sum_j Q_j \delta q_j &= \sum_i \sum_j \left[\frac{d}{dt}\left(m_i \vec{v}_i . \frac{\delta v_i}{\delta \dot{q}_j} \right) - m_i \vec{v}_i . \frac{\delta \vec{v}_i}{\delta q_j} \right] \delta q_j \\
&= \sum_j \left\{ \frac{d}{dt}\left[\frac{\delta}{\delta \dot{q}_j} \left(\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} \right) \right] - \frac{\delta}{\delta q_j} \left(\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} \right) \right\} \delta q_j \\
\sum_j Q_j \delta q_j &= \left[\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j} \right) - \frac{\delta T}{\delta q_j} \right]
\end{aligned}$$

where $T = \sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}$ is the kinetic energy of the system.

$$\sum_j \left[\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j} \right) - \frac{\delta T}{\delta q_j} - Q_j \right] \delta q_j = 0$$

Since all δq_j are independent whereas, $\delta \vec{r}_i$ were not. Thus,

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j} \right) - \frac{\delta T}{\delta q_j} - Q_j = 0 \tag{35}$$

When the forces F_i are derivable from a scalar potential function V

$$F_i = -\nabla_i V \tag{36}$$

Equation 36 in 31 gives

$$
\begin{aligned}
Q_j &= -\nabla_i V.\frac{\delta \vec{r}_i}{\delta q_j} \\
Q_j &= -\left(i\frac{\delta V}{\delta x_i} + j\frac{\delta V}{\delta y_i} + k\frac{\delta V}{\delta z_i}\right).\left(i\frac{\delta x_i}{\delta q_j} + j\frac{\delta y_i}{\delta q_j} + k\frac{\delta z_i}{\delta q_j}\right) \qquad (37)
\end{aligned}
$$

The equation 37 is exactly the same expression for the partial derives of a function $-V(r_l, r_2, ..., r_n; t)$ with respect to q_j.

$$
Q_j = -\frac{\delta V}{\delta q_j} \qquad (38)
$$

On combining the equations 35 and 38

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j} + \frac{\delta V}{\delta q_j} &= 0 \\
\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta(T - V)}{\delta q_j} &= 0 \qquad (39)
\end{aligned}
$$

The equations of motion in the form 39 are restricted to conservative systems, only when V is independent of time. Hence the potential V is a function of generalized coordinates q_j only, and does not depend upon $\dot{q}_j$. We define a new function, the Lagrangian L, as

$$
L(q_j, \dot{q}_j) = T(q_j, \dot{q}_j) - V(q_j) \qquad (40)
$$

The partial derives of the equation 40 are

$$\frac{\delta L}{\delta \dot{q}_j} = \frac{\delta T}{\delta \dot{q}_j} \tag{41}$$

$$\frac{\delta L}{\delta q_j} = \frac{\delta T}{\delta q_j} - \frac{\delta V}{\delta q_j} = \frac{\delta (T-V)}{\delta q_j} \tag{42}$$

Equations 41 and 42 in equation 39,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_j}\right) - \frac{\delta L}{\delta q_j} = 0, \qquad j = 1, 2, 3, ..., m. \tag{43}$$

Equation 43 gives set of m equations. These m equations, one for each independent generalized coordinates, are known as Lagrange's Equations of motion of the second kind in a potential field.

4. Examples

Kinetic energy T is given by

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i v_i^2 \\
&= \frac{1}{2}\sum_i m_i \left(\sum_{j=1}^{m} \frac{\delta \vec{r}_i}{\delta q_j}\dot{q}_j + \frac{\delta \vec{r}_i}{\delta t}\right)^2 \qquad \text{by using equation 3.28} \\
&= \frac{1}{2}\sum_i m_i \left[\sum_{j=1}^{m} \frac{\delta \vec{r}_i}{\delta q_j}\dot{q}_j \sum_{k=1}^{m} \frac{\delta \vec{r}_i}{\delta q_k}\dot{q}_k + 2\sum_{j=1}^{m} \frac{\delta \vec{r}_i}{\delta q_j}\dot{q}_j \frac{\delta \vec{r}_i}{\delta t} + \left(\frac{\delta \vec{r}_i}{\delta t}\right)^2\right] \\
&= \frac{1}{2}\sum_j \sum_k \sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta q_k}\dot{q}_j\dot{q}_k + \sum_j \sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta t}\dot{q}_j + \frac{1}{2}\sum_i m_i \left(\frac{\delta \vec{r}_i}{\delta t}\right)^2 \\
T &= \frac{1}{2}\sum_j \sum_k m_{jk}\dot{q}_j\dot{q}_k + \sum_j m_j \dot{q}_j + m_0 \qquad (44)
\end{aligned}$$

where

$$\begin{aligned}
m_{jk} &= \sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta q_k} \qquad (45) \\
m_j &= \sum_i m_i \frac{\delta \vec{r}_i}{\delta q_j}\frac{\delta \vec{r}_i}{\delta t} \\
m_0 &= \frac{1}{2}\sum_i m_i \left(\frac{\delta \vec{r}_i}{\delta t}\right)^2
\end{aligned}$$

Thus, the kinetic energy of a system can always be written as the sum of three homogeneous functions of the generalized velocities,

$$T = T_2 + T_1 + T_0 \tag{46}$$

where T_0 is independent of the generalized velocities, T, is linear in the velocities, and T_2 is quadratic in the velocities. If the transformation equations do not contain the time explicitly, as may occur when the constraints are independent of time (scleronomous),

$$m_0 = 0 \text{ and } m_j = 0 \quad \Longrightarrow \quad T_0 = 0 \text{ and } T_1 = 0.$$

Then

$$T = T_2 = \frac{1}{2} \sum_j \sum_k m_{jk} \dot{q}_j \dot{q}_k \tag{47}$$

Thus, T is always a homogeneous quadratic form in the generalized velocities.

1. **Motion of a single particle**

   (a) *Using Cartesian coordinates:* If (x, y, z) are the cartesian coordinates at time t of a free material point of mass m moving in a potential field $V(x, y, z)$, we may take $q_l = x, q_2 = y, q_3 = z$ as there are no equations of constraint. The applied force $\vec{F}$ on the particle has the components $-\frac{\delta V}{\delta x}, -\frac{\delta V}{\delta y}, -\frac{\delta V}{\delta z}$, while the kinetic energy T is given by $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Thus the Lagrangian for the particle is

$$T - V = L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \tag{48}$$

$$\frac{\delta L}{\delta \dot{x}} = m\dot{x} \tag{49}$$

$$\frac{\delta L}{\delta x} = -\frac{\delta V}{\delta x} \tag{50}$$

Equations 45 and 46 in equation 43,

$$\frac{d}{dt}(m\dot{x}) + \frac{\delta V}{\delta x} = 0$$

$$m\ddot{x} = -\frac{\delta V}{\delta x} = F_x$$

$$\text{Similarly} \quad m\ddot{y} = F_y, \qquad m\ddot{z} = F_z$$

The Lagrangian equations of motion are

$$m\ddot{x} = F_x, \qquad m\ddot{y} = F_y, \qquad m\ddot{z} = F_z \tag{51}$$

Equation 47 gives Newton's equations of motion.

(b) *Using cylindrical polar coordinates*: If (r, θ, z) are the cylindrical coordinates at time t of a free material point of mass m, we may take $(q_1 = r, q_2 = \theta, q_3 = z)$ as there are no equations of constraint. The applied force $\vec{F}$ on the particle has

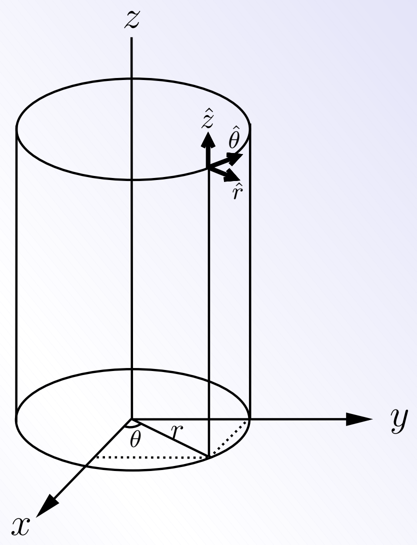

Figure 1: Cylindrical coordinates; $d\vec{r} = dr\,\hat{r} + rd\theta\,\hat{\theta} + dz\,\hat{k}$.

the components in generalized coordinates (Q_r, Q_θ, Q_z).

The three Lagrangian equations can be written by using the equation 35,

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{r}}\right) - \frac{\delta T}{\delta r} - Q_r = 0 \qquad (52)$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\theta}}\right) - \frac{\delta T}{\delta \theta} - Q_\theta = 0 \qquad (53)$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{z}}\right) - \frac{\delta T}{\delta z} - Q_z = 0 \qquad (54)$$

The position vector $\vec{r}$ in cylindrical coordinates,

$$\vec{r} = r\hat{r} + r\theta\hat{\theta} + z\hat{k}$$

where $\hat{r}, \hat{\theta}$ and $\hat{k}$ are unit vectors in the r, θ and z directions, respectively. The components of the force in generalized coordinates can be obtained from the equation 31 as,

$$Q_r = F_r \frac{\delta\vec{r}}{\delta r} = F_r \hat{r} \tag{55}$$

$$Q_\theta = F_\theta \frac{\delta\vec{r}}{\delta\theta} = F_\theta r\hat{\theta} \tag{56}$$

$$Q_z = F_z \frac{\delta\vec{r}}{\delta z} = F_z \hat{k} \tag{57}$$

In cylindrical coordinates

$$x = r cos\theta, \qquad y = r sin\theta, \qquad z = z$$

$$\frac{dx}{dt} = -r sin\theta \frac{d\theta}{dt} + \frac{dr}{dt} cos\theta, \qquad \frac{dy}{dt} = r cos\theta \frac{d\theta}{dt} + \frac{dr}{dt} sin\theta, \qquad \frac{dz}{dt} = \frac{dz}{dt}$$

$$\dot{x} = -r sin\theta \, \dot{\theta} + \dot{r} cos\theta, \qquad \dot{y} = r cos\theta \, \dot{\theta} + \dot{r} sin\theta, \qquad \dot{z} = \dot{z}$$

The kinetic energy T is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ &= \frac{1}{2}m(-rsin\theta\ \dot{\theta} + \dot{r}cos\theta)^2 + (rcos\theta\ \dot{\theta} + \dot{r}sin\theta)^2 + \dot{z}^2 \\ T &= \frac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2] \quad (58) \\ \frac{\delta T}{\delta \dot{r}} &= m\dot{r}, \qquad \frac{\delta T}{\delta r} = mr\dot{\theta}^2 \quad (59) \\ \frac{\delta T}{\delta \dot{\theta}} &= mr^2\dot{\theta}, \qquad \frac{\delta T}{\delta \theta} = 0 \quad (60) \\ \frac{\delta T}{\delta \dot{z}} &= m\dot{z}, \qquad \frac{\delta T}{\delta z} = 0 \quad (61) \end{aligned}$$

Equations 51 and 55 in equation 48,

$$\begin{aligned} \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - F_r &= 0 \\ m\ddot{r} - mr\dot{\theta}^2 &= F_r \quad (62) \end{aligned}$$

If r is constant, $F_r = -mr\dot{\theta}^2$ being the centripetal acceleration.

Equations 52 and 56 in equation 49,

$$\begin{aligned} \frac{d}{dt}(mr^2\dot{\theta}) &- rF_\theta = 0 \\ \frac{d\vec{L}}{dt} &= rF_\theta = N^l \quad (63) \end{aligned}$$

where $\vec{L} = mr^2\dot{\theta}$ is the angular momentum and N^l is the applied torque. Equations 53 and 57 in equation 50,

$$\begin{aligned} \frac{d}{dt}(m\dot{z}) \; - \; F_z = 0 \\ m\ddot{z} \; = \; F_z \end{aligned} \tag{64}$$

(c) *Using spherical polar coordinates*: If (r, θ, ϕ) are the spherical polar coordinates at time t of a free material point of mass m, we may take $(q_1 = r, q_2 = \theta, q_3 = \phi)$ as there are no equations of constraint. The applied force $\vec{F}$ on the particle has the components in generalized coordinates (Q_r, Q_θ, Q_ϕ).

The three Lagrangian equations can be written by using the equation 35,

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{r}}\right) - \frac{\delta T}{\delta r} - Q_r \; = \; 0 \tag{65}$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\theta}}\right) - \frac{\delta T}{\delta \theta} - Q_\theta \; = \; 0 \tag{66}$$

$$\frac{d}{dt}\left(\frac{\delta T}{\delta \dot{\phi}}\right) - \frac{\delta T}{\delta \phi} - Q_\phi \; = \; 0 \tag{67}$$

The position vector $\vec{r}$ in spherical coordinates,

$$\vec{r} \; = \; r\,\hat{r} + r sin\phi\theta\,\hat{\theta} + r\phi\,\hat{\phi}$$

where $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ are unit vectors in the r, θ and ϕ directions, respectively. The components of the force in generalized coordinates can be obtained from the

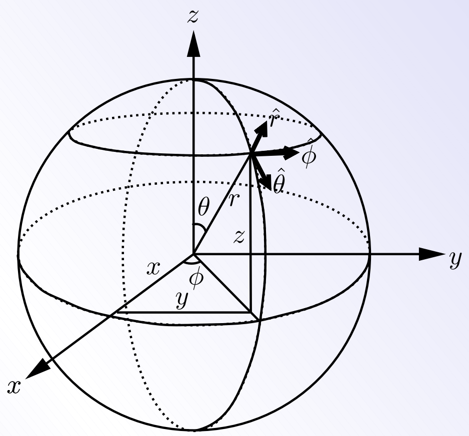

Figure 2: Spherical coordinates; $d\vec{r} = dr\,\hat{r} + rsin\phi d\theta\,\hat{\theta} + rd\phi\,\hat{\phi}$.

equation 31 as,

$$Q_r = F_r \frac{\delta \vec{r}}{\delta r} = F_r\ \hat{r} \quad (68)$$

$$Q_\theta = F_\theta \frac{\delta \vec{r}}{\delta \theta} = F_\theta r sin\phi\ \hat{\theta} \quad (69)$$

$$Q_\phi = F_\phi \frac{\delta \vec{r}}{\delta \phi} = F_\phi r\ \hat{\phi} \quad (70)$$

In spherical coordinates

$$x = rcos\theta\ sin\phi, \qquad y = rsin\theta\ sin\phi, \qquad z = rcos\phi$$
$$\dot{x} = \dot{r}cos\theta\ sin\phi - rsin\theta\ \dot{\theta}\ sin\phi + rcos\theta\ cos\phi\ \dot{\phi}$$
$$\dot{y} = \dot{r}sin\theta\ sin\phi + rcos\theta\ \dot{\theta}\ sin\phi + rsin\theta\ cos\phi\ \dot{\phi}$$
$$\dot{z} = \dot{r}cos\phi + rsin\phi\ \dot{\phi}$$

The kinetic energy T is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$T = \frac{1}{2}m\left[\dot{r}^2 + \left(rsin\phi\dot{\theta}\right)^2 + \left(r\dot{\phi}\right)^2\right] \tag{71}$$

$$\frac{\delta T}{\delta \dot{r}} = m\dot{r}, \qquad \frac{\delta T}{\delta r} = m\left(rsin^2\phi\dot{\theta}^2 + r\dot{\phi}^2\right) \tag{72}$$
$$\frac{\delta T}{\delta \dot{\theta}} = mr^2sin^2\phi\ \dot{\theta}, \qquad \frac{\delta T}{\delta \theta} = 0 \tag{73}$$
$$\frac{\delta T}{\delta \dot{\phi}} = mr^2\dot{\phi}, \qquad \frac{\delta T}{\delta \phi} = mr^2\dot{\theta}^2cos\phi \tag{74}$$

Equations 64 and 68 in equation 61,

$$\frac{d}{dt}(m\dot{r}) - m\left(rsin^2\phi\dot{\theta}^2 + r\dot{\phi}^2\right) - F_r = 0$$
$$m\ddot{r} - m\left(rsin^2\phi\dot{\theta}^2 + r\dot{\phi}^2\right) = F_r \tag{75}$$

Equations 65 and 69 in equation 62,

$$\begin{aligned}\frac{d}{dt}(mr^2sin^2\phi\dot{\theta}) &- rsin\phi\ F_\theta = 0\\ \frac{d}{dt}(mr^2sin^2\phi\dot{\theta}) &= rsin\phi\ F_\theta\end{aligned} \tag{76}$$

Equations 66 and 70 in equation 63,

$$\begin{aligned}\frac{d}{dt}(mr^2\dot{\phi}) - mr^2\dot{\theta}^2cos\phi &- rF_\phi = 0\\ \frac{d}{dt}(mr^2\dot{\phi}) - mr^2\dot{\theta}^2cos\phi &= rF_\phi\end{aligned} \tag{77}$$

2. **Atwood's machine:** Figure 3 shows the schematic diagram of Atwood's machine which is an example of a conservative system with holonomic, scleronomous constraint (the pulley is assumed frictionless and massless). Clearly there is only one independent coordinate y, the position of the other weight being determined by the constraint that the length of the rope between them is l. The Lagrangian equation for the motion can be written by using equation43 as,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{y}}\right) - \frac{\delta L}{\delta y} = 0 \tag{78}$$

The potential energy is

$$V = -M_1gy - M_2g(l-y) \tag{79}$$

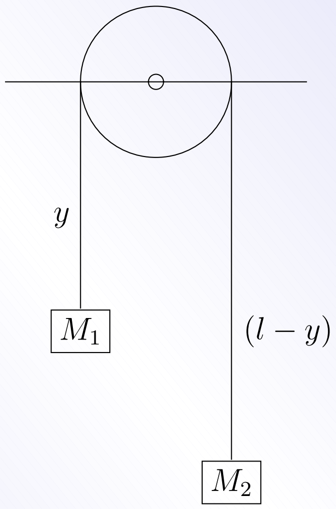

Figure 3: Atwood's machine.

The kinetic energy of the system is

$$T = \frac{1}{2}(M_1 + M_2)\dot{y}^2 \quad (80)$$

The Lagrangian L is

$$L = T - V = \frac{1}{2}(M_1 + M_2)\dot{y}^2 + M_1 g y + M_2 g(l - y) \tag{81}$$

$$L\frac{\delta L}{\delta \dot{y}} = (M_1 + M_2)\dot{y} \tag{82}$$

$$L\frac{\delta L}{\delta y} = M_1 g - M_2 g \tag{83}$$

Equations 78 and 79 in equation 74,

$$\begin{aligned} \frac{d}{dt}\left[(M_1 + M_2)\dot{y}\right] - M_1 g + M_2 g &= 0 \\ (M_1 + M_2)\ddot{y} &= (M_1 - M_2)g \\ \ddot{y} &= \frac{(M_1 - M_2)}{(M_1 + M_2)}g \end{aligned}$$

This is the familiar result obtained by more elementary means. This emphasizes that the forces of constraint here the tension in the rope appear nowhere in the Lagrangian formulation. Also the tension in the rope can not be found directly by the Lagrangian method.

3. **A bead (or ring) sliding on a uniformly rotating wire in a force-free space:** Consider beads in a straight wire, and is rotated uniformly with angular velocity $\vec{\omega}$ about some fixed axis perpendicular to the wire. This example has been chosen as a simple illustration of a constraint being time dependent, with the rotation axis along

z and the wire in the xy plane. Beads can move along wire, only one Lagrangian equation as using the equation 43,

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{r}}\right) - \frac{\delta L}{\delta r} = 0 \tag{84}$$

The transformation equations explicitly contain the time,

$$x = rcos\omega t, \qquad y = rsin\omega t$$
$$\dot{x} = -rsin\omega t\ \omega + \dot{r}cos\omega t, \qquad \dot{y} = rcos\omega t\ \omega + \dot{r}sin\omega t,$$

The kinetic energy T is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$
$$T = \frac{1}{2}m[\dot{r}^2 + (r\omega)^2] \tag{85}$$

The Lagrangian L is

$$T - V = L = \frac{1}{2}m[\dot{r}^2 + (r\omega)^2]$$
$$\frac{\delta L}{\delta \dot{r}} = m\dot{r} \tag{86}$$
$$\frac{\delta L}{\delta r} = mr\omega^2 \tag{87}$$

Equations 82 and 83 in equation 80,

$$\frac{d}{dt}(m\dot{r}) - mr\omega^2 = 0$$
$$\ddot{r} = r\omega^2 \tag{88}$$

The equation 84 is the familiar simple harmonic oscillator equation with a change of sign. The solution of the equation is $r = e^{\omega t}$ shows that the bead moves exponentially outward because of the centripetal acceleration. But the method cannot furnish the force of constraint that keeps the bead on the wire.

5. Velocity dependent potential

Consider an electric charge, q, of mass m moving at a velocity, v, in an electric field, $\vec{E}$, and a magnetic field, $\vec{B}$, which may depend upon time and position. The electric charge experiences a force, called the Lorentz force, given by

$$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] \tag{89}$$

Both $\vec{E}(t, x, y, z)$ and $\vec{B}(t, x, y, z)$ are continuous functions of time and position derivable from a scalar potential $\phi(t, x, y, z)$ and a vector potential $\vec{A}(t, x, y, z)$.
Faraday's law of electromagnetic induction is

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{\delta \vec{B}}{\delta t} = -\frac{\delta}{\delta t}(\nabla \times \vec{A}) = -\nabla \times \frac{\delta \vec{A}}{\delta t} \\
\nabla \times \vec{E} + \nabla \times \frac{\delta \vec{A}}{\delta t} &= 0 \\
\nabla \times \left(\vec{E} + \frac{\delta \vec{A}}{\delta t}\right) &= 0
\end{aligned}$$

Thus, $\vec{E} + \delta\vec{A}/\delta t$ is gradient of some scalar function ϕ. i.e.,

$$\begin{aligned}
\left(\vec{E} + \frac{\delta \vec{A}}{\delta t}\right) &= -\nabla\phi \\
\vec{E} &= -\nabla\phi - \frac{\delta \vec{A}}{\delta t} \qquad (90)
\end{aligned}$$

Equation 86 in 85,

$$\vec{F} = q\left[-\nabla\phi - \frac{\delta\vec{A}}{\delta t} + (\vec{v}\times\nabla\times\vec{A})\right] \tag{91}$$

Taking x component of $\vec{F}$,

$$\begin{aligned} F_x &= q\left[-\frac{\delta\phi}{\delta x} - \frac{\delta A_x}{\delta t} + (\vec{v}\times\nabla\times\vec{A})_x\right] \\ &= q\left[-\frac{\delta\phi}{\delta x} - \frac{\delta A_x}{\delta t} + v_y\left(\nabla\times A\right)_z + v_z\left(\nabla\times A\right)_y\right] \\ &= q\left[-\frac{\delta\phi}{\delta x} - \frac{\delta A_x}{\delta t} + v_y\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right) + v_z\left(\frac{\delta A_z}{\delta x} - \frac{\delta A_x}{\delta z}\right)\right] \end{aligned} \tag{92}$$

We can write

$$\begin{aligned} \frac{dA_x}{dt} &= \frac{\delta A_x}{\delta t} + \frac{\delta A_x}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta A_x}{\delta y}\frac{\delta y}{\delta t} + \frac{\delta A_x}{\delta z}\frac{\delta z}{\delta t} \\ -\frac{\delta A_x}{\delta t} &= -\frac{dA_x}{dt} + \frac{\delta A_x}{\delta x}v_x + \frac{\delta A_x}{\delta y}v_y + \frac{\delta A_x}{\delta z}v_z \end{aligned} \tag{93}$$

Equation 89 in 88,

$$\begin{aligned} F_x &= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta A_x}{\delta x}v_x + \frac{\delta A_x}{\delta y}v_y + \frac{\delta A_x}{\delta z}v_z\right] \\ &\quad + q\left[v_y\left(\frac{\delta A_y}{\delta x} - \frac{\delta A_x}{\delta y}\right) + v_z\left(\frac{\delta A_z}{\delta x} - \frac{\delta A_x}{\delta z}\right)\right] \qquad (94) \\ &= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta A_x}{\delta x}v_x + \frac{\delta A_y}{\delta x}v_y + \frac{\delta A_z}{\delta x}v_z\right] \\ &= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta}{\delta x}(A_x v_x + A_y v_y + A_z v_z)\right] \\ F_x &= q\left[-\frac{\delta\phi}{\delta x} - \frac{dA_x}{dt} + \frac{\delta}{\delta x}(\vec{A}.\vec{v})\right] \\ F_x &= q\left[-\frac{\delta}{\delta x}(\phi - \vec{A}.\vec{v}) - \frac{dA_x}{dt}\right] \qquad (95) \end{aligned}$$

We also can write

$$\begin{aligned} -\frac{dA_x}{dt} &= -\frac{d}{dt}\frac{\delta}{\delta v_x}(\vec{A}.\vec{v}) \\ -\frac{dA_x}{dt} &= \frac{d}{dt}\frac{\delta}{\delta v_x}(\phi - \vec{A}.\vec{v}) \qquad (96) \end{aligned}$$

Equation 92 in 91,

$$\begin{aligned} F_x &= q\left[-\frac{\delta}{\delta x}(\phi - \vec{A}.\vec{v}) + \frac{d}{dt}\frac{\delta}{\delta v_x}(\phi - \vec{A}.\vec{v})\right] \\ F_x &= -\frac{\delta U}{\delta x} + \frac{d}{dt}\frac{\delta U}{\delta v_x} \qquad (97) \end{aligned}$$

where $U = q(\phi - \vec{A}.\vec{v})$ is the velocity dependent potential. The Lagrange's equation was written as (equation 35),

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{q}_j}\right) - \frac{\delta T}{\delta q_j} &= Q_j \\ \frac{d}{dt}\left(\frac{\delta T}{\delta \dot{x}}\right) - \frac{\delta T}{\delta x} &= Q_x = F_x \qquad (98) \end{aligned}$$

On combining equations 93 and 94,

$$\begin{aligned} \frac{d}{dt}\frac{\delta}{\delta \dot{x}}(T-U) - \frac{\delta}{\delta x}(T-U) &= 0 \\ \frac{d}{dt}\frac{\delta L}{\delta \dot{x}} - \frac{\delta L}{\delta x} &= 0 \qquad (99) \end{aligned}$$

Similarly,

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{y}} - \frac{\delta L}{\delta y} = 0 \qquad (100)$$

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{z}} - \frac{\delta L}{\delta z} = 0 \qquad (101)$$

where $L = T - U = \frac{1}{2}mv^2 - q(\phi - \vec{A}.\vec{v})$ is the Lagrangian for a charged particle in electromagnetic field. Thus, even electromagnetic forces can be accommodated in Lagrange's formulation.

6. Hamilton's principle

The motion of a conservative system from its configuration at time t_1 to its configuration at time t_2 is such that the line integral between the time t_1 and t_2 of the Lagrangian of the system has a stationary value for the actual path of the motion.

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\ dt \tag{102}$$

$L = T - V$ is the Lagrangian. Since $\int L dt$ has the dimensions of *energy* $\times$ *time* called action, the principle is sometimes referred to as the principle of least action. The integral is called the action integral.

Then the variation of the action integral for fixed time t_1 and t_2 must be zero. i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t)\ dt = 0 \tag{103}$$

I has a stationary value relative to paths differing infinitesimally from the correct function $q(t)$. Let $\eta(t)$ be a continuous function with continuous first derivative and $\eta(t_1) = \eta(t_2) = 0$. We construct an another curve $q(t, \alpha)$ as

$$q(t, \alpha) = q(t, 0) + \alpha\eta(t) \tag{104}$$

Then, with different values of α we will get different paths. For $\alpha = 0$, equation 100 gives the curve $q(t, 0) = q(t)$. For simplicity, it is assumed that both the correct path

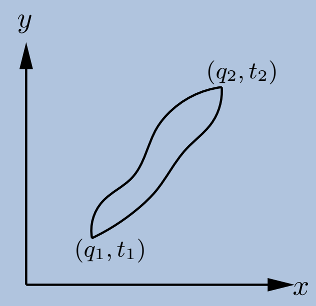

Figure 4: Varied paths of the function of $q(t)$ in the one-dimensional extremum problem.

$q(t)$ and the auxiliary function $\eta(t)$ are well-behaved functions and are continuous and nonsingular between t_1 and t_2, with continuous first and second derivatives in the same interval. For any such parametric family of curves, equation 98 can be written as,

$$I(\alpha) = \int_{t_1}^{t_2} L\{q(t,\alpha), \dot{q}(t,\alpha), t\}\; dt \tag{105}$$

and the condition for obtaining a stationary point is,

$$\left(\frac{dI}{d\alpha}\right)_{\alpha=0} \;=\; 0$$

Consider the equation 98,

$$\begin{aligned}
I &= \int_{t_1}^{t_2} L(q,\dot{q},t)\ dt \\
\frac{\delta I}{\delta\alpha}d\alpha &= \int_{t_1}^{t_2}\left(\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha + \frac{\delta L}{\delta\dot{q}}\frac{\delta\dot{q}}{\delta\alpha}d\alpha\right)\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \int_{t_1}^{t_2}\frac{\delta L}{\delta\dot{q}}\frac{\delta\dot{q}}{\delta\alpha}d\alpha\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \int_{t_1}^{t_2}\frac{\delta L}{\delta\dot{q}}\frac{\delta^2 q}{\delta\alpha\delta t}d\alpha\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \int_{t_1}^{t_2}\frac{\delta L}{\delta\dot{q}}\frac{\delta}{\delta t}\left(\frac{\delta q}{\delta\alpha}\right)d\alpha\ dt \\
&= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt + \left.\frac{\delta L}{\delta\dot{q}}\frac{\delta q}{\delta\alpha}\right|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\delta q}{\delta\alpha}\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{q}}\right)d\alpha\ dt
\end{aligned}$$

$$\text{Since}\quad \left(\frac{\delta q}{\delta\alpha}\right)_{t_1} = \left(\frac{\delta q}{\delta\alpha}\right)_{t_2} = 0,$$

$$\begin{aligned}
\frac{\delta I}{\delta\alpha}d\alpha &= \int_{t_1}^{t_2}\frac{\delta L}{\delta q}\frac{\delta q}{\delta\alpha}d\alpha\ dt - \int_{t_1}^{t_2}\frac{\delta q}{\delta\alpha}\frac{d}{dt}\left(\frac{\delta L}{\delta\dot{q}}\right)d\alpha\ dt \\
\frac{\delta I}{\delta\alpha}d\alpha &= \int_{t_1}^{t_2}\left(\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta\dot{q}}\right)\frac{\delta q}{\delta\alpha}d\alpha\ dt
\end{aligned}$$

$$\text{At}\quad \alpha = 0, \left(\frac{\delta \mathrm{q}}{\delta\alpha}\right)_0 \mathrm{d}\alpha = \delta\mathrm{q}\quad\text{and}\quad \left(\frac{\delta I}{\delta\alpha}\right)_0 d\alpha = \delta I$$

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} \right) \delta q \; dt$$

According to Hamilton's principle $\delta I = 0$,

$$\int_{t_1}^{t_2} \left(\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} \right) \delta q \; dt = 0 \qquad (106)$$

Since the q variables are independent, the variations δq are independent. In the equation 102, the coefficients of δq must vanish separately.

$$\frac{\delta L}{\delta q} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}} = 0 \qquad (107)$$

The equation 103 is the Lagrange equation of motion. This equation is valid for any function $f(q, \dot{q}, t)$ and is called the Euler-Lagrange differential equation.