Classical Mechanics

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1. Introduction

Moving coordinate systems are important because, no material body is at absolute rest. As we know, even galaxies are not stationary. Therefore, a coordinate frame at absolute rest is hypothetical, hypothesized by Newton, where his laws of motion hold. In reality, we have the moving frames, prime example being Earth itself. We therefore need to know how the Newton's laws operate in a moving frame like a rotating frame (e.g. Earth).

2. Rectilinear moving frame

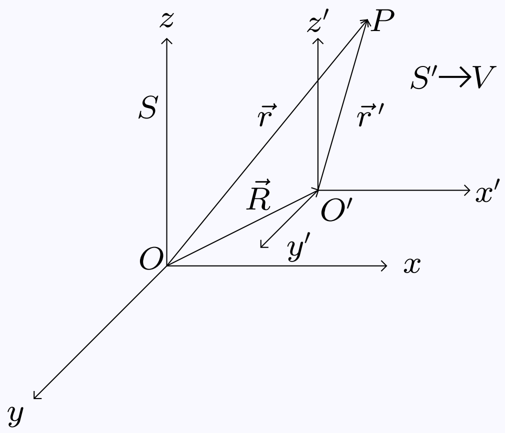

Figure 1: Frame S', moving relative to S along vector $R(t)$.

Let us consider a simplest case of a rectilinear motion. Let the frame S' move relative to S. For simplicity, let us assume that, the axes of S and S' are parallel. The instantaneous position of O' relative to the origin O is depicted by $\vec{R}(t)$ in figure 1. Consider a rectilinear motion of S' along the direction of $\vec{R}(t)$. Then vector $\vec{r}$ in S will be $\vec{r}\,'$ in S' such that,

$$\vec{r}\,' = \vec{r} - \vec{R}(t) \tag{1}$$

The time derivative of the equation is

$$\dot{\vec{r}}\,' = \dot{\vec{r}} - \dot{\vec{R}}(t) \tag{2}$$

$\vec{v}\,'$ is the velocity in S' and is related with velocity $\vec{v}$ in S as,

$$\vec{v}\,' = \vec{v} - \vec{V} \tag{3}$$

where $V = \dot{\vec{R}}(t)$ is velocity of S' relative to S. Let us assume that, the Newton's second law is valid in S. Then, the external force F_{ext} is

$$\begin{aligned} F_{ext} &= m\ddot{\vec{r}} = m\frac{d\vec{v}}{dt} \\ &= m\left(\frac{d\vec{v}\,'}{dt} + \frac{d\vec{V}}{dt}\right) = m\left(\ddot{\vec{r}}\,' + \frac{d\vec{V}}{dt}\right) \end{aligned} \tag{4}$$

If S' moves relative to S with constant velocity, i.e., $dV/dt = 0$, then S and S' are indistinguishable.

$$F_{ext} = m\ddot{\vec{r}} = m\ddot{\vec{r}}\,' \tag{5}$$

Thus, S and S' are inertial. We call the inertial force measured in the fixed frame S as $F = m\ddot{\vec{r}}$. Then, the inertial force measured in S' is

$F' = m\ddot{\vec{r}}'$. The F_{ext} is independent of motion of S and S'. If $\dot{\vec{V}} \neq 0$,

$$F' \;=\; m\ddot{\vec{r}} - m\ddot{\vec{R}} = F - m\ddot{\vec{R}} \tag{6}$$

It must be noted that if S' is moving with respect to S with constant velocity, the law of force has a same form. (In technical jargon, it is covariant). However, if S' is an accelerated frame, then it can be distinguished from S by virtue of the term $m\ddot{\vec{R}}$. Some times, the term $m\ddot{\vec{R}}$ of S' is called fictitious force. This is a misnomer because, we get a jerk when a train starts.

3. Rotating frame of reference

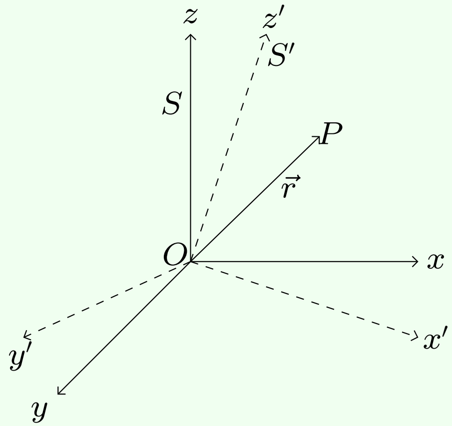

Figure 2: Vector $\bar{r}$ in S and S'.

Consider two frame of reference S and S' with unit vectors $n = (i, j, k)$ and $n' = (i', j', k')$ and with a common origin. S' rotates with some axis with angular velocity $\vec{\omega}$. Let us consider a position vector $\vec{r}$. The components of $\vec{r}$ in S are (x, y, z) and in S' are (x', y', z'). Then,

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \quad (7)$$

$$\vec{r} = \hat{i}'x' + \hat{j}'y' + \hat{k}'z' \quad (8)$$

As S' rotates, unit vectors $n' = (i', j', k')$ are functions of time. Let us write d/dt as a time derivative operator in S and d'/dt as a time derivative operator in S'. We do not write d'/dt' because we assume $t = t'$, a non relativistic case. Since S is a fixed frame, (i, j, k) do not change with time. Clearly,

$$v = \frac{d\vec{r}}{dt} = \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z} \quad (9)$$

The time derivative of the equation 9 is

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= \hat{i}'\frac{dx'}{dt} + \frac{d\hat{i}'}{dt}x' + \hat{j}'\frac{dy'}{dt} + \frac{d\hat{j}'}{dt}y' + \hat{k}'\frac{dz'}{dt} + \frac{d\hat{k}'}{dt}z' \\
\frac{d\vec{r}}{dt} &= \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt} + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt} \\
v &= v' + x'\frac{d\hat{i}'}{dt} + y'\frac{d\hat{j}'}{dt} + z'\frac{d\hat{k}'}{dt} \quad (10)
\end{aligned}$$

where

$$v' = \hat{i}'\frac{dx'}{dt} + \hat{j}'\frac{dy'}{dt} + \hat{k}'\frac{dz'}{dt}$$

is the velocity in the frame S'. We need to find the value of dn'/dt in a fixed frame in terms of the known quantities like $\vec{\omega}$ and $\vec{r}$. We must prove a following important lemma.

Lemma: Let there be a general direction OM around which a vector A of constant magnitude rotates with a constant angular velocity $\vec{\omega}$ in a fixed frame. Then

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$$

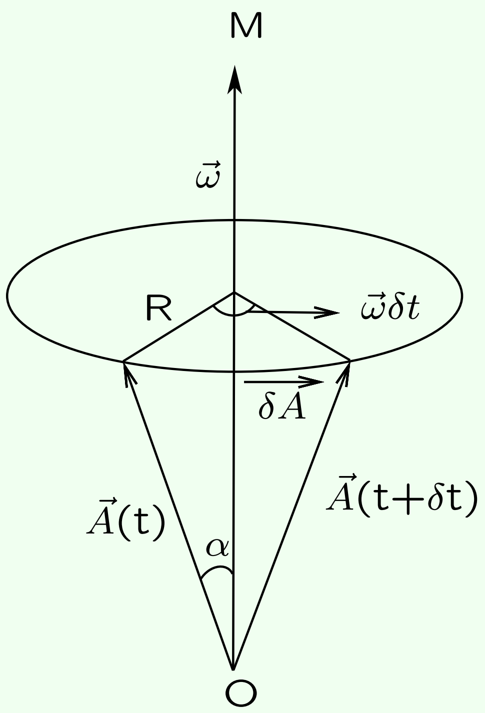

Figure 3: Vector $\vec{A}$ is rotating around a fixed direction OM.

Proof: Let the tail of the vector A be at O and let it rotate about OM with angular velocity $\vec{\omega}$ as shown in the figure 3. The position of the vector A at time t and $t + \delta t$ is shown in the figure when it rotates in time δt. As the direction of $\vec{A}$ changes, $A(\vec{t})$ changes to $\vec{A}(t + \delta t)$. The magnitude of $\vec{A}(t)$ and $\vec{A}(t + \delta t)$ is same but,

$$\vec{A}(t + \delta t) = \vec{A}(t) + \delta \vec{A} \tag{11}$$

From the figure 3,

$$\mid \delta A \mid = R \, \omega \, \delta t \tag{12}$$

Again form the figure 3,

$$R = A \, sin \, \alpha \tag{13}$$

$$\begin{aligned} \delta \vec{A} &= \omega A \, sin \, \alpha \, \delta t \\ \delta \vec{A} &= (\vec{\omega} \times \vec{A}) \, \delta t \\ \frac{\delta \vec{A}}{\delta t} &= \vec{\omega} \times \vec{A} \end{aligned} \tag{14}$$

When $\delta t \to 0$, the equation 14 gives

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A} \tag{15}$$

This derivative is taken in the frame S. In the frame where A is fixed (i.e., S' frame), $d'A/dt = 0$. And, in S', OM rotates with angular velocity $-\vec{\omega}$ i.e., in a counter clockwise direction. Equation (1.15) can be written in an operator form as,

$$\frac{d(\,)}{dt} = \vec{\omega} \times (\,) \tag{16}$$

where, ( ) contain any vector operator. Thus,

$$\frac{d\hat{i}'}{dt} = \vec{\omega} \times \hat{i}' \tag{17}$$

$$\frac{d\hat{j}'}{dt} = \vec{\omega} \times \hat{j}' \tag{18}$$

$$\frac{d\hat{k}'}{dt} = \vec{\omega} \times \hat{k}' \tag{19}$$

Using equations 17, 18 and 19 in equation 10,

$$v = v' + x'(\vec{\omega} \times \hat{i}') + y'(\vec{\omega} \times \hat{j}') + z'(\vec{\omega} \times \hat{k}')$$

$$v = v' + \vec{\omega} \times (\hat{i}'x' + \hat{j}'y' + \hat{k}'z')$$

$$v = v' + \vec{\omega} \times \vec{r} \tag{20}$$

$$\frac{d\vec{r}}{dt} = \frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r} \tag{21}$$

From equation 21, the general operator equation can be written as,

$$\frac{d(\,)}{dt} = \frac{d'(\,)}{dt} + \vec{\omega} \times (\,) \tag{22}$$

We use equation 22 to get acceleration in a rotating frame in terms of its value in the fixed frame. We denote a and a' as accelerations in S and S'. Thus by operating $\vec{v}$ to the equation 22,

$$\frac{d\vec{v}}{dt} = \frac{d'\vec{v}}{dt} + \vec{\omega} \times \vec{v} \tag{23}$$

$$\begin{aligned}
\frac{dv}{dt} &= \frac{d'}{dt}(\vec{v}\,' + \vec{\omega} \times \vec{r}) + \vec{\omega} \times (\vec{v}\,' + \vec{\omega} \times \vec{r}) \\
a &= \frac{d'\vec{v}\,'}{dt} + \vec{\omega} \times \frac{d'\vec{r}}{dt} + \frac{d'\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \vec{v}\,' + \vec{\omega} \times \vec{\omega} \times \vec{r} \\
a &= a' + 2(\vec{\omega} \times v') + \vec{\omega} \times \vec{\omega} \times \vec{r} + \frac{d'\vec{\omega}}{dt} \times \vec{r}
\end{aligned} \tag{24}$$

It must be noted that, the time derivative of a vector parallel to ω is same in both S and the S' frames. Thus,

$$\frac{d'\vec{\omega}}{dt} = \frac{d\vec{\omega}}{dt}$$

If F_{ext} is the external force, according to Newton's second law of motion, valid in the fixed frame as,

$$F_{ext} = ma \tag{25}$$

Using equation 24,

$$F_{ext} = ma = ma' + 2m(\vec{\omega} \times v') + m(\vec{\omega} \times \vec{\omega} \times \vec{r}) + m(\dot{\vec{\omega}} \times \vec{r}) \tag{26}$$

As everyone is on the surface of Earth, everyone is in a rotating frame, he measures $\vec{v}\,'$ and $\vec{a}\,'$ and not v and a. Equation 26 for an observer in rotating frame as,

$$ma' = F_{ext} - 2m(\vec{\omega} \times v') - m(\vec{\omega} \times \vec{\omega} \times \vec{r}) - m(\dot{\vec{\omega}} \times \vec{r}) \tag{27}$$

Equation 27 gives a motion of a particle in the rotating frames. If ω is constant, and it is the case in many situations, the last term $-m(\dot{\vec{\omega}} \times \vec{r} = 0)$, then,

$$ma' = F_{ext} - 2m(\vec{\omega} \times v') - m(\vec{\omega} \times \vec{\omega} \times \vec{r}) \tag{28}$$

$$F' = F_{ext} + F_c + F_r = F_{ext} + F_o \tag{29}$$

where $F_c = 2m(\vec{\omega} \times v')$ is called Coriolis force and $F_r = -m(\vec{\omega} \times \vec{\omega} \times \vec{r})$ is called centrifugal force. In a fixed frame, the term $m(\vec{\omega} \times \vec{\omega} \times \vec{r})$ is called a centripetal force, directed towards the center and its magnitude is $m\omega^2 r = mv^2/r$. In a rotating frame, we have a minus sign in front of the centripetal force, directed outwards and is called centrifugal force.

$F_o = F_c + F_r$ is not real force. It is called fictitious force. It is not present in a fixed coordinate system. We can treat a rotating coordinate system as if it were fixed by adding a centrifugal force and a Coriolis force. If we fix a coordinate system with the rotating particle, then it is at rest in this frame. The centripetal force is balanced by centrifugal force in this frame. Coriolis force depends on the velocity of the particle and it acts in a direction perpendicular to $\vec{v}$. Therefore, Coriolis force does no work but only changes its direction of motion.

Exercise: Estimate the magnitude of

1. Angular velocity of earth
2. Centrifugal acceleration at equator
3. Coriolis acceleration at latitude of 45^o with velocity $10^3 ms^{-1}$

4. Applications

4.1. Effect of centrifugal force on acceleration due to gravity

The acceleration due to gravity varies with latitude ϕ, being about 0.5% smaller at equator than its value at the poles. This is the reason why our Earth has oblate shape i.e., Earth's sphere of flattened at the poles.

Consider a point P with latitude ϕ as shown in figure 4. If Earth was not rotating, g at P would act along PO. Centrifugal acceleration is to be superposed on g to get g_{eff}. Thus,

$$g_{eff} = g - \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{30}$$

Direction of g_{eff} does not pass through O, the centre of Earth. The direction of plumb line is along g_{eff} as shown in figure 4. Earth's surface is flattened to such an extent that is always perpendicular to g_{eff}

4.2. Effect of Coriolis force on atmospheric air flow

It is observed that, there is not much of an air flow in vertical direction as compared with a horizontal direction. This is so because, in a vertical

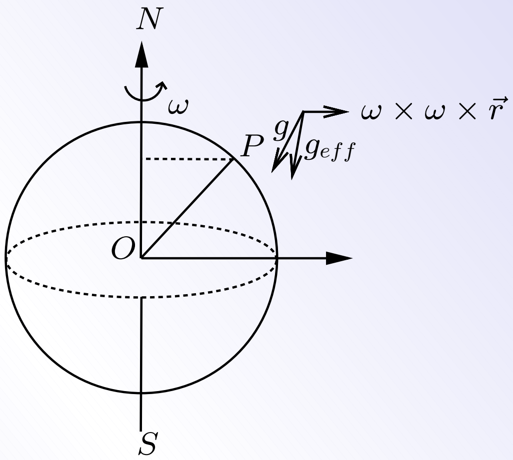

Figure 4: Direction of effective acceleration due to gravity

direction, pressure decrease as one moves upwards and this force is balanced by weight of the air parcel. This leads to a persistent long range motion of air in the form of winds.

In the absence of Coriolis force, the air flow takes place from higher pressure to lower pressure. Let ω be perpendicular to the plane. The Coriolis acceleration $(-2\omega \times v)$ deflects air current as shown in figure 5. The wind current then circulates around a low pressure zone in clockwise direction.

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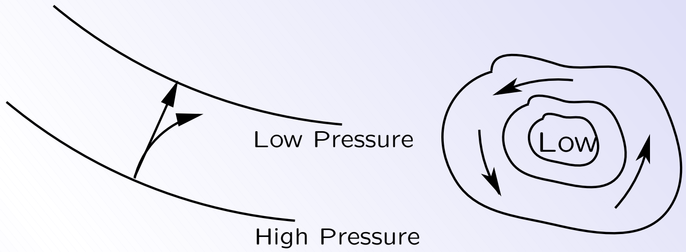

Figure 5: Cyclone formation due to Coriolis force

4.3. Foucault pendulum

French physicist Foucault realized that, the Coriolis force would rotate the plane of oscillation of pendulum. Foucault pendulum is an ordinary pendulum with a large length as much as more than 10 meter.

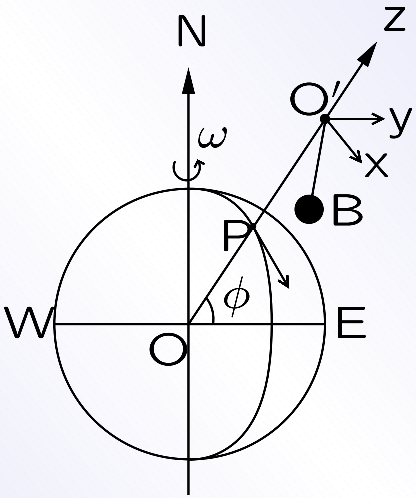

Figure 6: Position of the Foucault's pendulum on the surface of the Earth.

Let us assume that a pendulum is suspended from the origin O' of a frame S' at a place of latitude ϕ, as shown in figure 6. The $Z-axis$ of S' is vertically upward, the $X-axis$ due south and the $Y-axis$ due east, but with the origin at a height above the surface of the earth at P. The position vector of the bob B is $\vec{r} \equiv (x, y, z)$ and $O'B = l = \vec{r}$ is the length of the pendulum. The tension in the string is the applied force F_a and since it is along BO', we may write,

$$F_a = k_1^2 m\vec{r}$$
$$\ddot{\vec{r}} = k_1^2\vec{r} \quad (31)$$

where $k_1^2 = g/l$ is a constant of proportionality. Therefore, the equation of motion of the pendulum is,

$$\ddot{\vec{r}} - k_1^2\vec{r} = 0 \quad (32)$$

The equation 31 can be satisfied only when there is no other force acting on the pendulum except gravitational force. If the rotation of Earth is taken into account, $g > g_{eff}$ and in addition a Coriolis force act on the pendulum. Thus, the equation of pendulum becomes,

$$\ddot{\vec{r}} - k^2\vec{r} = -2\vec{\omega} \times \vec{v} \quad (33)$$

where $k^2 = g_{eff}/l$.

Since x-axis due south and the Y-axis due to east, the vector $\vec{\omega}$ is in the ZX-plane, so that $\omega_y = 0$ and we have

$$\omega_x = -\omega cos\phi, \quad \omega_y = 0 \quad \omega_z = \omega sin\phi, \tag{34}$$

Since motion of the pendulum is in the XY-plane, so that $v_z = 0$. Then clearly

$$\vec{\omega} = (-\omega cos\phi, 0, \omega\, sin\phi) \tag{35}$$

$$\ddot{\vec{r}} = (\ddot{x}, \ddot{y}, 0) \tag{36}$$

Resolving the equation of motion in components,

$$\ddot{x} - k^2 x = -2(\vec{\omega} \times \vec{v})_x = 2\omega\dot{y}\, sin\phi \tag{37}$$

$$\ddot{y} - k^2 y = -2(\vec{\omega} \times \vec{v})_y = -2\omega(\dot{x}\, sin\phi + \dot{z}\, cos\phi) \tag{38}$$

$$\ddot{z} - k^2 z + g = -2(\vec{\omega} \times \vec{v})_z = 2\omega\dot{y}\, cos\phi \tag{39}$$

For small oscillations, $z = -l$ and therefore $\dot{z} = \ddot{z} = 0$. Again, since $g \approx 1000\,\text{cms}^{-2}$ and $\omega = 7.3 \times 10^{-5} rad/s$, the term containing ω in equation 39 may be neglected in comparison with g and therefore we obtain $k^2 = -g/l$.

Then equations 37 and 38 may be written as

$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\, sin\phi \tag{40}$$
$$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x}\, sin\phi \tag{41}$$

Introducing the complex variable $z = x+iy$ and writing $u = \omega sin\phi$, equations 40 and 41 may be written as a single equation

$$\ddot{z} + 2iu\dot{z} + \frac{g}{l}z = 0 \tag{42}$$

[1]

Solution of the equation 42 is in the form $z = e^{i\alpha t}$, where α is a constant and on substitution to the equation 42,

$$\alpha^2 + 2u\alpha - \frac{g}{l} = 0 \tag{43}$$

The roots of the equation 43 are,

$$\alpha_1 = -u + \sqrt{u^2 + \frac{g}{l}} \quad \alpha_2 = -u - \sqrt{u^2 + \frac{g}{l}} \tag{44}$$

1

$$z = x + iy, \quad \dot{z} = \dot{x} + i\dot{y}, \quad \ddot{z} = \ddot{x} + i\ddot{y} = -\frac{g}{l}x + 2\omega\dot{y}\, sin\phi - i\frac{g}{l}y - i2\omega\dot{x}\, sin\phi$$
$$\ddot{z} = -\frac{g}{l}(x + iy) - 2i\omega\, sin\phi(\dot{x} + i\dot{y}) = -\frac{g}{l}z - 2i\omega\, sin\phi\, \dot{z}$$

Thus general solution of the equation 42 is

$$z = Ae^{i\alpha_1 t} + Be^{i\alpha_2 t} \tag{45}$$
$$\dot{z} = i\alpha_1 Ae^{i\alpha_1 t} + i\alpha_2 Be^{i\alpha_2 t} \tag{46}$$

where A and B are constants to be determined by the initial conditions. If the pendulum is drawn due south (along the X-axis) at distance *a* and let go with zero initial velocity in the ZX-plane, we have, as initial conditions, $z = (x + iy) = (a + 0) = a, \dot{z} = 0$ at $t = 0$. Equation 45 and 46 yeilds,

$$a = A + B \tag{47}$$
$$0 = \alpha_1 A + \alpha_2 B, \quad -\alpha_1 A = \alpha_2 B \tag{48}$$

Then from equation 45,

$$\begin{aligned}\dot{z} &= iA\alpha_1(e^{\alpha_1 t} - e^{i\alpha_2 t}) \\ &= iA\alpha_1\left[e^{i\left(-u+\sqrt{u^2+\frac{g}{l}}\right)t} - e^{i\left(-u-\sqrt{u^2-\frac{g}{l}}\right)t}\right] \\ &= iA\alpha_1 e^{-iut}\left[e^{i\left(\sqrt{u^2+\frac{g}{l}}\right)t} - e^{-i\left(\sqrt{u^2-\frac{g}{l}}\right)t}\right]\end{aligned}$$
$$\dot{z} = -2A\alpha_1 e^{-iut} sin\left(\sqrt{u^2 + \frac{g}{l}}t\right) \tag{49}$$

As we know, the velocity of the bob becomes zero every time the pendulum reverses its direction and at these values of *t*, we have $\dot{z} = 0$, and this happens when the sine function is zero and therefore for values of t for which

$$\sqrt{u^2 + \frac{g}{l}}t = n\pi; \quad (n = 0, 1, 2, 3, ...)$$

For one oscillation $t = T$ and $n = 2$,

$$\sqrt{u^2 + \frac{g}{l}}T = 2\pi; \quad T = \frac{2\pi}{\sqrt{u^2 + \frac{g}{l}}} \tag{50}$$

where $u = \omega\, sin\phi$. If ω is negligible, $u \approx 0$ and we obtain $T = 2\pi g/l$. If $u \neq 0$,

$$\sqrt{u^2 + \frac{g}{l}} = \frac{2\pi}{T} \tag{51}$$

Using equation 44 in 51,

$$\alpha_1 = -u + \frac{2\pi}{T} \quad \alpha_2 = -u - \frac{2\pi}{T} \tag{52}$$

Rewriting the equation 45 using 52,

$$\begin{aligned} z(t) &= Ae^{i\left(-u+\frac{2\pi}{T}\right)t} + Be^{i\left(-u-\frac{2\pi}{T}\right)t} \\ z(t) &= e^{-iut}\left(Ae^{\frac{i2\pi t}{T}} + Be^{\frac{-i2\pi t}{T}}\right) \end{aligned} \tag{53}$$

When the pendulum is initially set in motion by drawing it southward along the X-axis to the position P_1 with a distance *a*, we have $t = 0$ and therefore, we see from equation 53 that

$$z(0) \;=\; A + B = a \tag{54}$$

After half an oscillation, $t = T/2$, the bob is at the northern end at P_2 and we have

$$z(T/2) \;=\; e^{-iuT/2}(-A - B) = -ae^{-iuT/2} = ae^{i(\pi - uT/2)} \tag{55}$$

Equation 55 shows that, the vector $Z(0)$ representing the point P_1 has been rotated into the position P_2 through an angle $(\pi - uT/2)$. This means that the trajectory of the pendulum, in the first half oscillation is the curve from P_1 to P_2 as shown in figure 7. Similarly, when $t = T$, we have

$$z(T) \;=\; e^{-iuT}(A + B) = ae^{-iuT} = z(0)e^{-iuT} \tag{56}$$

Equation 56 shows that after one complete oscillation, in time T, the position of the bob of the pendulum is again at the southern end but rotated through an angle $-uT$ into position P_3. Thus the trajectory in the second half oscillation is the curved path from P_2 to P_3 as shown in the figure. We thus see that the motion of the bob is as if the plane of vibration of the pendulum rotated clockwise through an angle uT in one period. Thus the time required for one complete rotation of the plane,

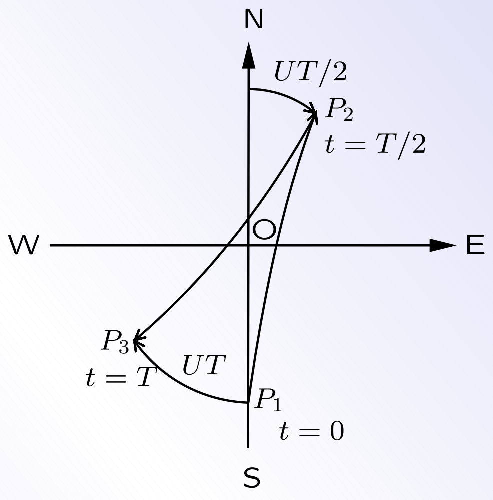

Figure 7: Rotation of the plane of oscillation of Foucault's pendulum.

ie., though an angle 2π is

$$\tau \;=\; \frac{T}{uT}2\pi = \frac{2\pi}{\omega\, sin\phi} = \frac{24}{sin\phi}\text{hours} \tag{57}$$

since $\frac{2\pi}{\omega} = 1day = 24$ hours. Thus, at a place of latitude $\phi = 45^o$, the time τ for one complete rotation is $24\sqrt{2} \approx 34$ hours.

We note that the curved path from P_1 to P_2 of the bob of the pendulum is due to the force of Coriolis acting to the right of the velocity vector as the bob moves from south to north. For the same reason, the force of Coriolis acting to the right of the velocity as the bob moves from north to south accounts for the part of the trajectory from P_2 to P_3. We recall that the other inertial force, namely the centrifugal force has already been taken care of in defining, the weight of a material point so that the trajectory P_1 to P_2 to P_3 of the bob is solely due to the force of Coriolis. Foucault carried out his experiments in 1851 in Paris, but his experiments only confirmed the rotation of the earth qualitatively. Quantitative confirmation came only in the year 1879 by the work of Kamerlingh Onnes of low temperature fame.

5. Conservation of linear momentum and angular momentum

Consider a system of *n* particles of masses $m_1, m_2, m_3, ..., m_n$ at respective positions $r_1(t)$, $r_2(t)$, $r_3(t)$, ... , $r_n(t)$ at a time *t*. Then, the total momentum $\vec{P}$ of the system is,

$$\vec{P} = \sum_{i=1}^{n} \vec{p_i} = \sum_{i=1}^{n} m_i \vec{v_i} = \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \vec{r_i}\right) \tag{58}$$

The force acting on the i^{th} particle of the system has two parts (i) external force $F_i^{(e)}$, (from our side) and (ii) internal force, F_{ji} (internal force on the i^{th} particle due to the j^{th} particle). Thus, the equation of motion (Newton's second law) for the i^{th} particle is written as

$$\frac{d\vec{p_i}}{dt} = \sum_{j=1}^{n-1} \vec{F}_{ji} + \vec{F}_i^{(e)} \tag{59}$$

$j \neq i$, because a particle cannot exert any force on itself. Clearly, from equation 59, the total force acting on the system is

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}\sum_i \vec{p_i} = \sum_i \sum_j \vec{F}_{ji} + \sum_i \vec{F}_i^{(e)} \tag{60}$$

If the Newton's third law is valid in the system, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\sum_i \sum_j \vec{F}_{ji} = 0$, the equation 60 becomes,

$$\begin{aligned} \frac{d\vec{P}}{dt} &= \frac{d}{dt}\sum_i \vec{p}_i = \vec{F}_i^{(e)} = F_{ext} \qquad (61) \\ &= \frac{d}{dt}\sum_i m_i\vec{v}_i = \frac{d^2}{dt^2}\sum_i m_i\vec{r}_i = \sum_i \vec{F}_i^{(e)} = F_{ext} \\ &= \frac{d^2}{dt^2}\sum_i m_i\vec{r}_i = \sum_i \vec{F}_i^{(e)} = F_{ext} \qquad (62) \end{aligned}$$

We define a vector $\vec{R}$ as the average of the radii vectors of the particles, weighted in proportion to their mass,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots\ldots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \ldots\ldots + m_n} = \frac{\sum_i m_i\vec{r}_i}{M} \qquad (63)$$

The vector $\vec{R}$ defines a point known as the *center of mass*. The equation 62 reduces to

$$\frac{d\vec{P}}{dt} = M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = F_{ext} \qquad (64)$$

The equation 62 states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. If $F_{ext} = 0$, the total linear momentum of the system,

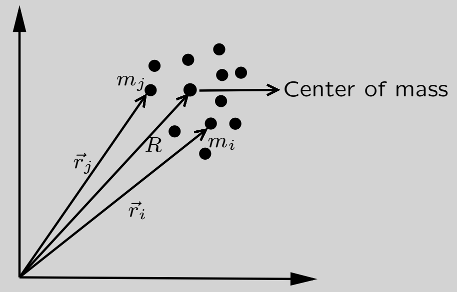

Figure 8: The center of mass of a system of particles.

$$\frac{d\vec{P}}{dt} = M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = 0 \tag{65}$$

$$\vec{P} = M\frac{d\vec{R}}{dt} = Constant$$

That is total linear momentum of the system (*total mass of the system times the velocity of the center of mass*) is constant. Thus the conservation theorem for the linear Momentum of a system of particles is stated as, **if the total external force acting on the system is zero, the total linear momentum is conserved**.

The angular momentum of the system of particles is

$$L = \sum_i \vec{r}_i \times \vec{p}_i \tag{66}$$

The torque acting on the system is

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) \\ &= \sum_i \frac{d\vec{r}_i}{dt} \times \vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} \\ &= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} + \sum_i \vec{r}_i \times \sum_j^{n-1} \vec{F}_{ij} \qquad (67)\end{aligned}$$

If the Newton's third law is valid in the system, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\sum_i \sum_j \vec{F}_{ij} = 0$, the equation 67 becomes,

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \sum_i \vec{r}_i \times \vec{F}_i^{(e)} \\ \frac{d\vec{L}}{dt} &= \sum_i \vec{N}_i^{(e)} = N_{ext} \qquad (68)\end{aligned}$$

If $N_{ext} = 0$, the total angular momentum of the system is

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \sum_i \vec{N}_i^{(e)} = 0 \\ \vec{L} &= Constant\end{aligned}$$

That is total angular momentum of the system is constant. Thus the conservation theorem for the angular momentum of a system of particles is stated as **the angular momentum of the system is constant in time, if the applied (external) torque is zero**.

5.1. Theorem:

Angular momentum of a system of particles about a general origin O is equal to the angular momentum of the system concentrated at the CM plus the angular momentum of the system about its CM.

Proof: Consider a particle whose coordinate is $\vec{r}_i$ with respect to O and $\vec{r}_i{}'$ with respect to the CM. Thus

$$\vec{r}_i = \vec{R} + \vec{r}_i{}' \quad (69)$$

$$\vec{v}_i = \vec{V}_{CM} + \vec{v}_i{}' \quad (70)$$

$$m_i\vec{v}_i = m_i\vec{V}_{CM} + m_i\vec{v}_i{}'$$

$$\vec{p}_i = m_i\vec{V}_{CM} + \vec{p}_i{}' \quad (71)$$

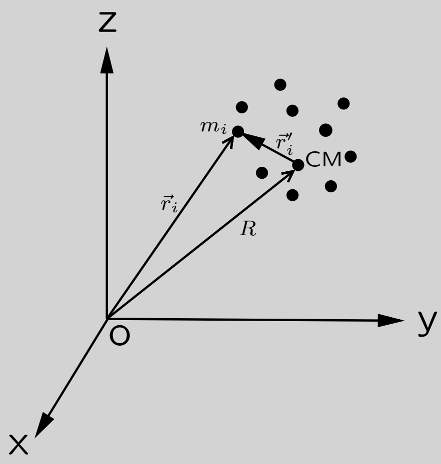

Figure 9: Position of i^{th} particle.

where, $\vec{p}_i{}' = m_i\vec{v}_i{}'$ is a velocity of i^{th} particle with reference to the *CM*. Then,

$$\begin{aligned} \vec{L} &= \sum_i \vec{r}_i \times \vec{p}_i \\ &= \sum_i \left(\vec{R} + \vec{r}_i{}'\right) \times \left(m_i\vec{V}_{CM} + \vec{p}_i{}'\right) \\ \vec{L} &= \vec{R} \times \vec{V}_{CM} \sum_i m_i + \vec{R} \times \sum_i \vec{p}_i{}' + \\ & \sum_i (m_i\vec{r}_i{}') \times \vec{V}_{CM} + \sum_i (\vec{r}_i{}' \times \vec{p}_i{}') \end{aligned} \tag{72}$$

If CM is taken taken as the origin,

$$\sum_i (m_i \vec{r}_i{}') = 0 = \sum_i \vec{p}_i{}' \qquad (73)$$

The equation 72 reduces to,

$$\vec{L} = \vec{R} \times \vec{V}_{CM} \sum_i m_i + \sum_i (\vec{r}_i{}' \times \vec{p}_i{}')$$
$$\vec{L} = \vec{R} \times \vec{P} + \sum_i (\vec{r}_i{}' \times \vec{p}_i{}') \qquad (74)$$

$\vec{R} \times \vec{P}$ is an angular momentum of the centre of mass with respect to O, and $\sum_i (\vec{r}_i{}' \times \vec{p}_i{}')$ is an angular momentum of the system of particles with respect to the *CM*. From the above equation it is obvious that, $\vec{L}$ depends upon the choice of origin. If the *CM* of the system of particles is stationary, then $\vec{L}$ is independent of the choice of origin.

6. Energy

The work done by all forces in moving the system from an initial configuration *1*, to a final configuration *2* is

$$\begin{aligned} \int_1^2 W_{12} dW &= \sum_i \int_1^2 \vec{F}_i . d\vec{r}_i \\ \int_1^2 W_{12} dW &= \sum_i \int_1^2 \vec{F}_i^{(e)} . d\vec{r}_i + \sum_i \sum_j^{n-1} \int_1^2 \vec{F}_{ij} . d\vec{r}_{ij} \end{aligned} \tag{75}$$

If the Newton's third law is valid in the system, $\vec{F}_{ij} = -\vec{F}_{ji}$, then $\sum_i \sum_j \vec{F}_{ij} = 0$, the equation 75 becomes,

$$\begin{aligned} T_2 - T_1 &= \sum_i \int_1^2 \vec{F}_i^{(e)} . d\vec{r}_i \\ &= \sum_i \int_1^2 \frac{d\vec{p}_i}{dt} . \frac{d\vec{r}_i}{dt} dt \\ &= \sum_i \frac{1}{m_i} \int_1^2 \frac{d\vec{p}_i}{dt} . \vec{p}_i dt \\ &= \sum_i \frac{1}{m_i} \int_1^2 \vec{p}_i \, d\vec{p}_i \end{aligned}$$

$$T_2 - T_1 = \left[\sum_i \frac{\vec{p_i}^{\,2}}{2m_i}\right]_2 - \left[\sum_i \frac{\vec{p_i}^{\,2}}{2m_i}\right]_1$$

$$T = \frac{1}{2}\sum_i m_i v_i^2 \qquad (76)$$

In center of mass coordinates, $\vec{v_i} = \vec{V}_{CM} + \vec{v_i}\,'$ and $\vec{v_i}^{\,2} = (\vec{V}_{CM} + \vec{v_i}\,').(\vec{V}_{CM} + \vec{v_i}\,')$,

$$T = \frac{1}{2}\sum_i m_i(\vec{V}_{CM} + \vec{v_i}\,').(\vec{V}_{CM} + \vec{v_i}\,')$$

$$= \frac{1}{2}\sum_i m_i\vec{V}_{CM}^2 + \vec{V}_{CM}\sum_i m_i\vec{v_i}\,' + \frac{1}{2}\sum_i m_i\left(\vec{v_i}\,'\right)^2$$

If CM is taken taken as the origin, $\sum_i \vec{p_i}\,' = \sum_i m_i\vec{v_i}\,' = 0$,

$$T = \frac{1}{2}M{V_{CM}}^2 + \frac{1}{2}\sum_i m_i\left(\vec{v_i}\,'\right)^2 \qquad (77)$$

Thus the kinetic energy of a system of particles is equal to the sum of the kinetic energy of the *CM* plus the kinetic energy of the system about its *CM*.

If the particle moves from initial configuration *1*, to a final configuration *2* under the action of a conservative force, then the external forces are derivable in terms of the gradient of a potential, the first term of the equation 75 can be written as [2]

$$\sum_i \int_1^2 \vec{F}_i^{(e)} d\vec{r}_i = -\sum_i \int_1^2 \nabla_i V_i d\vec{r}_i \tag{78}$$

The internal forces are also conservative, then the mutual forces between the i^{th} and j^{th} particles, F_{ij} and F_{ji} can be obtained from a potential function V_{ij}. The seond term of the equation 75 can be written as

$$\sum_i \sum_j \int_1^2 \vec{F}_{ij} d\vec{r}_{ij} = -\sum_i \sum_j \int_1^2 (\nabla_i V_{ij} d\vec{r}_i + \nabla_j V_{ij} d\vec{r}_j) \tag{79}$$

To satisfy the strong law of action and reaction, $\vec{F}_{ij} = \nabla_j V_{ij} = -\nabla_i V_{ij} = -\vec{F}_{ji}$. Then, equation 79 can be written as,

$$\begin{aligned} \sum_i \sum_j \int_1^2 \vec{F}_{ij} d\vec{r}_{ij} &= -\sum_i \sum_j \int_1^2 \nabla_i V_{ij} (d\vec{r}_i - d\vec{r}_j) \\ &= -\sum_i \sum_j \int_1^2 \nabla_{ij} V_{ij} d\vec{r}_{ij} \end{aligned} \tag{80}$$

Where $\vec{r}_{ij} = d\vec{r}_i - d\vec{r}_j$ and ∇_{ij} stands for the gradient with respect to $\vec{r}_{ij}$. The term $\nabla_{ij} V_{ij} d\vec{r}_{ij}$ in the equation 80 is for the ij pair of particles. The total work arising from internal forces then reduces to

$$\sum_i \sum_j \int_1^2 \vec{F}_{ij} d\vec{r}_{ij} = -\sum_i \sum_j \int_1^2 \frac{1}{2} \nabla_{ij} V_{ij} d\vec{r}_{ij} \tag{81}$$

[2] If $\vec{F}$ is conservative, $\nabla \times \vec{F} = 0$ then $\vec{F} = \nabla V$

Combining the equation 81, equation 78 and equation 75, we see that,

$$\begin{aligned}\int_1^2 W_{12} dW &= -\sum_i \int_1^2 \nabla_i V_i d\vec{r}_i - \sum_i \sum_j \int_1^2 \frac{1}{2} \nabla_{ij} V_{ij} d\vec{r}_{ij} \\ -V &= -\sum_i [(\nabla_i V_i)_2 - (\nabla_i V_i)_1] - \frac{1}{2} \sum_i \sum_j [(\nabla_{ij} V_{ij})_2 - (\nabla_{ij} V_{ij})_1] \\ V &= \sum_i \nabla_i V_i + \frac{1}{2} \sum_i \sum_j \nabla_{ij} V_{ij} \qquad (82)\end{aligned}$$

The second term on the right in equation 82 will be called the internal potential energy of the system. In general, it need not be zero and, more important, it may vary as the system changes with time. Only for the particular class of systems known as rigid bodies the internal potential always be constant.

7. Constraints

Constraints means restrictions; constrained motion means restricted motion. Most of the motion that we encounter, is constrained motion. Most physical realizations of constrained motion involve surfaces of other bodies, for example,

1. **Motion of a billiard ball on the table:** Motion of a billiard ball is restricted by the boundaries of the table, and it moves on the surface of the table. If the centre of mass of a billiard ball of radius R moving on a billiard table of length $2a$ and breadth $2b$, must satisfy the relation

$$-a + R \leq x \leq a - R, \qquad -b + R \leq y \leq b - R, \qquad z = R$$

assuming that the origin of the coordinate axes is at the centre of the rectangular table and x and y axes are parallel to length and breadth respectively. i.e., a set of one equation and two inequalities, defines the motion of a billiard ball at all instants of time.

2. **The motion of a simple pendulum:** The bob of the pendulum moves in a vertical plane (say zx plane). Its distance from the fulcrum

is fixed. Thus, if (x, y, z) is coordinate of the bob then,

$$y = constant, \qquad z^2 + x^2 = l^2$$

are the restrictions on the coordinates of the bob.

Physically, constrained motion is realised by the forces which arise when the object in motion is in contact with the constraining surfaces or curves. These forces, called constraint forces, are usually stiff elastic forces at the contact. If there are no constraints, motion of the particle is described by the trajectory $\vec{r}(t) = ix + jy + kz$ and by its momentum $\vec{p}(t) = ip_x + jp_y + kp_z$. Thus the position of the particle is specified by three coordinates. If there are *N* particles, *3N* independent coordinates are necessary for the position specification of the system at a time *t*. Presence of constraints may reduce the number of independent variables.

7.1. Classification of Constraints

1. (a) **Scleronomic:** constraint relations do not explicitly depend on time,

   (b) **Rheonomic:** constraint relations depend explicitly on time,

2. (a) **Holonomic:** conditions of constraint can be expressed as equations connecting the coordinates of the particles,

   (b) **Non holonomic:** constraint relations are not holonomic,

3. (a) **Conservative:** total mechanical energy of the system is conserved while performing, the constrained motion. Constraint forces do not do any work,

   (b) **Dissipative:** constraint forces do work and total mechanical energy is not conserved.

4. (a) **Bilateral:** at any point on the constraint surface both the forward and backward motions are possible. Constraint relations are not in the form of inequalities but are in the form of equations,

   (b) **Unilateral:** at some points no forward motion is possible. Constraint relations are expressed in the form of inequalities.

7.2. Holonomic and non holonomic constraints

If one can write the equations of constraints as

$$
\begin{aligned}
f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
. & \\
. & \\
f_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
f_{i+1}(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
. & \\
. & \\
f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0
\end{aligned} \tag{83}
$$

where $k < n$, then such constraints are known as holonomic constraints. The constraints which cannot be expressed in the form of algebraic equations are non holonomic constraints, however, they could be expressed as inequalities.

7.3. Examples of constraints

1. **Rigid body:** A rigid body is a system of particles such that the distance between any pair of particles remains constant in time. Thus the motion of a rigid body is constrained by the equations

$$\vec{r}_i - \vec{r}_k = const. \tag{84}$$

where the pair of subscripts (i , k) run over all distinct pairs of particles forming the body. Obviously this constraint is scleronomic. The constraint is also holonomic and bilateral. The constraint relations 84 can be written as

$$|\vec{r}_i - \vec{r}_k|^2 = const.$$

Taking differentials

$$(\vec{r}_i - \vec{r}_k).\Delta(\vec{r}_i - \vec{r}_k) = 0 \tag{85}$$

Work done by the system is

$$W = \sum_i \sum_k (\vec{F}_{ik}.\Delta\vec{r}_i + \vec{F}_{ki}.\Delta\vec{r}_k) \tag{86}$$

Let the internal force of constraint on the i^{th} particle due to the k^{th} particle be represented by $\vec{F}_{ik}$. By Newton's third law we have,

$$\vec{F}_{ik} = -\vec{F}_{ki} \tag{87}$$

Thus we have for the work done by $\vec{F}_{ik}$ due to a displacement $\Delta\vec{r}_i$ of the i^{th} particle,

$$\vec{F}_{ik}.\Delta\vec{r}_i = -\vec{F}_{ki}.\Delta\vec{r}_k \tag{88}$$

On combining equations 86 and 88 we can write the total work done by the system

$$W = \sum_i \sum_k \vec{F}_{ik}.(\Delta\vec{r}_i - \Delta\vec{r}_k) \tag{89}$$

Since all $\vec{F}_{ik}$ are the internal forces which arise purely due to interaction between all possible pairs of particles, it is only natural that $\vec{F}_{ik}$ will act parallel to the line joining the i^{th} and k^{th} particles. Thus we can write,

$$\vec{F}_{ik} = C_{ik}(\vec{r}_i - \vec{r}_k) \tag{90}$$

where C_{ik}'s are real constants and symmetric in *i* and *k*. Substituting in the above expression for the total work, we have

$$W = \sum_i \sum_k C_{ik}(\vec{r}_i - \vec{r}_k).(\Delta\vec{r}_i - \Delta\vec{r}_k) \tag{91}$$

In equation 91 each individual term of the summand is zero. Thus the constraint of rigidity is conservative in nature, apart from its being scleronomic, holonomic and bilateral.

2. **Deformable bodies:** Suppose that the deformation of the body is changing in time according to a certain prescribed function of time. Then the motion of such a body is constrained by the equation

$$|\vec{r}_i - \vec{r}_k| = f(t) \tag{92}$$

where $\vec{r}_i$ and $\vec{r}_k$ are position vectors and the pair of subscripts (i , k) runs over all distinct pairs of particles in the body. These constraint relations cannot give the total work $W = 0$. Hence this is a rheonomic, holonomic, bilateral and dissipative constraint.

3. **Gas in a spherical container of radius R**. Here if $\vec{r}_i$ is a position vector of the i^{th} gas molecule (origin is at the centre of the sphere) then

$$x_i^2 + y_i^2 + z_i^2 \leq R^2 \tag{93}$$

Thus, we have a constraint equation given by an inequality and hence is non holonomic constraint.

4. **Rolling without sliding:** Suppose a spherical ball is rolling on a plane without sliding. We assume that the surfaces in contact are perfectly rough. Thus the frictional forces are not negligible. Since the point of contact is not sliding, the frictional forces do not do any work, and

hence the total mechanical energy of the rolling body is conserved. Thus the constraint is conservative. To obtain the constraint equation we note that rolling without sliding means that the relative velocity of the point of contact with respect to the plane is zero. Then the velocity v of any point P in the rolling body, as seen from a fixed frame of reference, is given by

$$v = V_{CM} + \vec{\omega} \times \vec{r} \tag{94}$$

where V_{CM} is the velocity of the centre of mass and $\vec{r}$ is measured from the *CM* to the point *P* under consideration. Thus the velocity of the point of contact is obtained by putting $\vec{r} = -r\hat{n}$ in equation 94, where $\hat{n}$ is the unit vector along the outward normal to the plane and r is the radius of the sphere. Since there is no sliding of this point we must have the instantaneous velocity v at the contact

$$v = V_{CM} - r(\vec{\omega} \times \hat{n}) = 0 \tag{95}$$

For a sphere this constraint is non integrable because ω is generally not expressible in the form of a total time derivative of any single coordinate. Thus the constraint is non holonomic. However, for a cylinder, $\omega = d\theta/dt$ where θ is the angle of rotation of the cylinder about its axis. Therefore this equation of constraint can be integrated and reduced to a holonomic form, giving a relation between r and the coordinates of the centre of mass.

8. Generalized coordinates

The problem of system of n particles can be solved when the number of constraint equations are less than $3n$. Let there be k equations of constraints $k < 3n$,

$$
\begin{aligned}
f_1(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
f_2(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \\
. & \\
. & \\
. & \\
. & \\
f_k(\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n; t) &= 0 \qquad (96)
\end{aligned}
$$

i.e., $3n - k$ coordinates may be regarded as free and which define the position of the system at any moment of time t. Then the number of independent coordinate to specify the motion at a given time t is $3n - k$. These independent coordinates are called *degrees of freedom*.

In the case of a free material particle, for instance, $n = 1$ and $k = 0$ so that it has $3n - k = 3$ degrees of freedom. If the particle is constrained to move on a surface whose equation may be taken as $f(x, y, z) = 0(z = 0)$, we clearly have $k = 1$ and therefore it would have $3n - k = 2$ degrees of freedom. On the other hand, for a dumb-bell shaped structure, with two particles connected by a rod of length l; a constraint equation becomes

$$f(x, y, z) = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - l^2 = 0$$

we have $n = 2$ and $k = 1$, therefore it has $3n - k = 5$ degrees of freedom. The degrees of freedom are represented by $3n - k$ variables, $q_1, q_2, , q_{n-k}$. The old coordinates $\vec{r}_1, \vec{r}_2, \vec{r}_3, ...\vec{r}_n$ are expressed in terms of $q's$ as,

$$\begin{aligned} \vec{r}_1 &= \vec{r}_1(q_1, q_2, , q_{n-k}; t) \\ \vec{r}_2 &= \vec{r}_2(q_1, q_2, , q_{n-k}; t) \\ &. \\ &. \\ &. \\ &. \\ \vec{r}_n &= \vec{r}_n(q_1, q_2, , q_{n-k}; t) \end{aligned} \tag{97}$$

8.1. Virtual displacement

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\vec{r}_i$, consistent with the forces and constraints imposed on the system at the given instant of time *t*. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval *dt*, during which the forces and constraints may be changing.

8.2. Virtual work

Total work done by the external forces when virtual displacements are made in *n* particle system, is known as virtual work. If $\vec{F}_i^{(a)}$ be the applied force and $\vec{f}_i$ be the constraint force acting on i_{th} particle, the net force acting on the system is

$$\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i \tag{98}$$

If $\delta\vec{r}_i$ is the virtual displacement, the work done on the system is

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = \sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i \tag{99}$$

When system is in equilibrium

$$W = \sum_i \vec{F}_i.\delta\vec{r}_i = 0 \tag{100}$$

Thus equation 99 reduces to

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i + \sum_i \vec{f}_i.\delta\vec{r}_i = 0 \tag{101}$$

The virtual displacements $\delta\vec{r}_i$ are such that the constraint forces do no work ($\sum_i \vec{f}_i.\delta\vec{r}_i = 0$). Thus,

$$\sum_i \vec{F}_i^{(a)}.\delta\vec{r}_i = 0 \tag{102}$$

i.e., The condition for static equilibrium is that the virtual work done by all the applied forces should vanish, provided the virtual work done by all the constraint forces vanishes. This is called the principle of virtual work.

8.3. D'Alembert's Principle

Consider the motion of *n* particle system. Then, by Newton's law,

$$\sum_i \vec{F}_i = \sum_i \dot{\vec{p}}_i \tag{103}$$

Combining equations 103 and 98,

$$\begin{aligned}\sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i &= \sum_i \dot{\vec{p}}_i \\ \sum_i \vec{F}_i^{(a)} + \sum_i \vec{f}_i - \sum_i \dot{\vec{p}}_i &= 0 \end{aligned} \quad (104)$$

The equation 104 states that the particles in the system will be in equilibrium under a force equal to the actual force plus a *reversed effective force* $-\dot{\vec{p}}_i$. The work done now can be written as

$$\sum_i (\vec{F}_i^{(a)} + \vec{f}_i - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \quad (105)$$

The virtual displacements $\delta\vec{r}_i$ are such that the constraint forces do no work ($\sum_i \vec{f}_i.\delta\vec{r}_i = 0$). Thus,

$$\sum_i (\vec{F}_i^{(a)} - \dot{\vec{p}}_i).\delta\vec{r}_i = 0 \quad (106)$$

The equation 106 is called D'Alembert's Principle. D'Alembert's principle does not involve forces of constraint. i.e., any dynamical problem could be converted into an effective static problem.